PAPER-1(B.E./B. TECH.)

JEE (Main) 2021

Questions & Solutions

(Reproduced from memory retention)

Date : 26 February, 2021 (SHIFT-2) Time ; (3.00 pm to 06.00 pm)

Duration : 3 Hours | Max. Marks : 300

SUBJECT : MATHEMATICS

MATHEMATICS

1. If $0 < a, b < 1$, $\tan^{-1}a + \tan^{-1}b = \frac{\pi}{4}$ find value of $(a+b) - \left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^3+b^3}{3}\right)$ ….. ∞

(1) $\ell n\ 3$ (2) $2\ \ell n\ 2$ (3) $\ell n\ 2$ (4) $\ell n\ 2 + \ell n\ 3$

Ans. (3)

Sol. $\tan^{-1}\left(\frac{a+b}{1-ab}\right) = \frac{\pi}{4} \Rightarrow a + b = 1 - ab \Rightarrow (1 + a)(1 + b) = 2$

Now, $a + b - \left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^3+b^3}{3}\right) \ldots\ldots \infty$

$= \left(a - \frac{a^2}{2} + \frac{a^3}{3} \ldots\ldots\right) + \left(b - \frac{b^2}{2} + \frac{b^3}{3} \ldots\ldots\right)$

$= \ell n(1 + a) + \ell n(1 + b) = \ell n(1 + a)(1 + b) = \ell n\ 2$

2. If $f(x) = \int_0^x e^t f(t)\, dt + e^x$, then f(x) is equal to

(1) $2e^{e^x-1}$ (2) $2e^{e^x-1}+1$ (3) $e^{e^x-1}+1$ (4) $2e^{e^x-1}-1$

Ans. (4)

Sol. $f'(x) = e^x . f(x) + e^x$

$\Rightarrow \frac{f'(x)}{f(x)+1} = e^x \Rightarrow \ell n(f(x) + 1) = e^x + c$

put $x = 0$

$\ell n\ 2 = 1 + c$

$\therefore \ell n(f(x) + 1) = e^x + \ell n\ 2 - 1$

$\Rightarrow f(x) + 1 = 2 . e^{e^x-1} \Rightarrow f(x) = 2e^{e^x-1} - 1$

3. A seven digit number has been formed by using digit 3, 3, 4, 4, 4, 1, 1 (by taking all at a time). Probability that number is even.

(1) $\frac{2}{7}$ (2) $\frac{3}{7}$ (3) $\frac{5}{14}$ (4) $\frac{3}{14}$

Ans. (2)

Sol. $n(S) = \frac{7!}{2!3!2!}$

$n(E) = \frac{6!}{2!2!2!}$

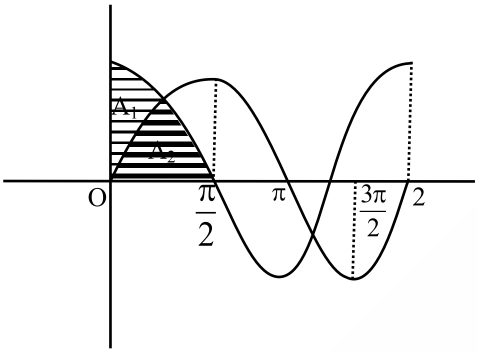
$P(E) = \frac{n(E)}{n(S)} = \frac{6!}{7!} \times \frac{2!3!2!}{2!2!2!}$

$= \frac{1}{7} \times 3 = \frac{3}{7}$

4. If A_1 & A_2 are area bounded by :
A_1 : y = sin x, y = cos x & y-axis in I[st] quadrant
A_2 : y = sin x, y = cos x & x-axis & $x = \frac{\pi}{2}$.
Then
(1) $A_1 : A_2 = 1 : \sqrt{2}$; $A_1 + A_2 = 1$
(2) $A_1 : A_2 = \sqrt{2} : 1$; $A_1 + A_2 = \sqrt{2} + 1$
(3) $A_1 : A_2 = 1 : 2$; $A_1 + A_2 = 2$
(4) $A_1 : A_2 = 1 : 2$; $A_1 + A_2 = 1$

Ans. (1)

Sol. $A_1 + A_2 = \int_0^{\pi/2} \cos x \, . dx = \sin x\Big|_0^{\pi/2} = 1$



$A_1 = \int_0^{\pi/4} (\cos x - \sin x)\, dx = (\sin x + \cos x)\Big|_0^{\pi/4} = \sqrt{2} - 1$

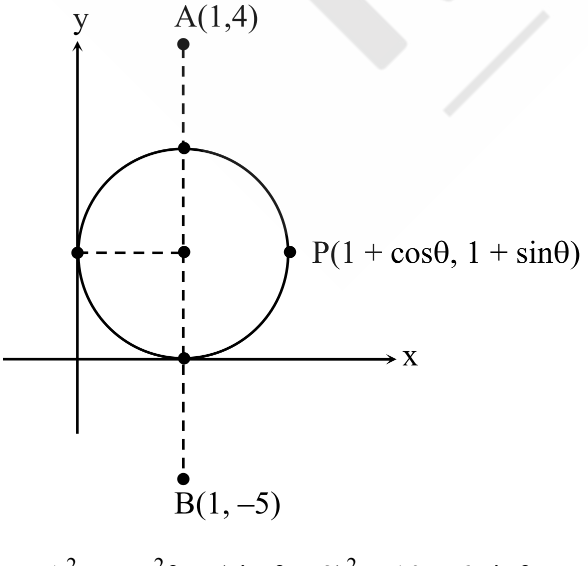
$\therefore\ A_2 = 1 - (\sqrt{2} - 1) = 2 - \sqrt{2}$

$\therefore\ \frac{A_1}{A_2} = \frac{\sqrt{2}-1}{\sqrt{2}(\sqrt{2}-1)} = 1 : \sqrt{2}$

5. Consider the circle $(x - 1)^2 + (y - 1)^2 = 1$; A(1, 4) B(1, –5). If P is a point on circle. Such that (PA) + (PB) is maximum, then P, A, B lie on ?
(1) ellipse (2) hyperbola (3) Straight line (4) None of these

Ans. (3)

Sol.



$\therefore\ PA^2 = \cos^2\theta + (\sin\theta - 3)^2 = 10 - 6\sin\theta$

$PB^2 = \cos^2\theta + (\sin\theta - 6)^2 = 37 - 12\sin\theta$

$PA^2 + PB^2 = 47 - 18\sin\theta|_{max.} \Rightarrow \theta = \frac{3\pi}{2}$

$\therefore$ P, A, B lie on a line x = 1

6. If A triangle is inscribed in a circle of radius r, then which of the following triangle can have maximum area

(1) equilateral triangle with height $\frac{2r}{3}$

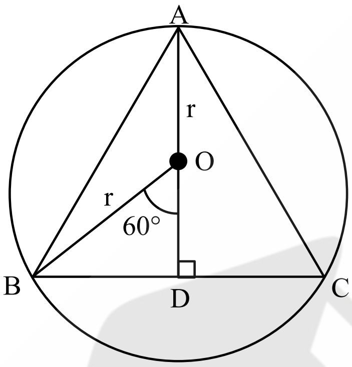
(2) right angle triangle with side 2r, r

(3) equilateral triangle with side $\sqrt{3}$r

(4) isosceles triangle with base 2r

Ans. (3)

Sol.



$OD = r\cos 60° = \frac{r}{2}$

$\text{Height} = AD = \frac{3r}{2}$

$\text{Now } \sin 60° = \frac{3\frac{r}{2}}{AB}$

$\Rightarrow AB = \sqrt{3}r$

7. If f'(a) = 2 and f(a) = 4, then find value of $\lim\limits_{x\to a}\frac{xf(a)-af(x)}{x-a}$

(1) 4 – 2a (2) 2a – 4 (3) 0 (4) a – 4

Ans. (1)

Sol. By L–H rule

$L = \lim\limits_{x\to a}\frac{f(a)-af'(x)}{1}$

$\therefore$ L = 4 – a (2)

8. If $\vec{a}_1 = x\hat{i}-\hat{j}+\hat{k}$, $\vec{a}_2 = \hat{i}+y\hat{j}+z\hat{k}$ are collinear vector, then a unit vector which is parallel to $x\hat{i}+y\hat{j}+z\hat{k}$, can be

(1) $\frac{1}{\sqrt{3}}(\hat{i}-\hat{j}+\hat{k})$ (2) $\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ (3) $\frac{1}{\sqrt{2}}(\hat{j}-\hat{k})$ (4) $\frac{\hat{i}-\hat{j}-\hat{k}}{\sqrt{3}}$

Ans. (1)

Sol. $\frac{x}{1} = -\frac{1}{y} = \frac{1}{z} = \lambda(\text{let})$

Unit vector parallel to $x\hat{i}+y\hat{j}+z\hat{k} = \pm\frac{\left(\lambda\hat{i}-\frac{1}{\lambda}\hat{j}+\frac{1}{\lambda}\hat{k}\right)}{\sqrt{\lambda^2+\frac{2}{\lambda^2}}}$

for $\lambda = 1$ it is $\pm\frac{(\hat{i}-\hat{j}+\hat{k})}{\sqrt{3}}$

9. If $y + z = 5$, $\frac{1}{y}+\frac{1}{z}=\frac{5}{6}, y > z$

If prime factorization of a natural number $N = 2^x\ 3^y\ 5^z$. Find number of odd divisors of N including 1.

(1) 6x (2) 6 (3) 12 (4) 11

Ans. (3)

Sol. Solving given two equation we get y = 3, z = 2

$\Rightarrow N = 2^x\ 3^3\ 5^2$

number of odd divisor = (2 + 1) (3 + 1) = 12

10. If a curve y = f(x) in given by $\frac{dy}{dx}=\frac{xy^2+y}{x}$ passing through (–2, 3) meets the line L = 0 at (3, y) then y is

(1) $\frac{-11}{19}$ (2) $\frac{-18}{19}$ (3) $\frac{-11}{29}$ (4) $\frac{11}{19}$

Ans. (2)

Sol. $\frac{dy}{dx}=\frac{xy^2+y}{x}$

$\Rightarrow \frac{xdy-ydx}{y^2} = x\,dx$

$\Rightarrow -d\left(\frac{x}{y}\right) = d\left(\frac{x^2}{2}\right)$

$\Rightarrow \frac{-x}{y} = \frac{x^2}{2} + C$

It passes through (–2, 3)

$\Rightarrow \frac{2}{3} = 2 + C$

$\Rightarrow C = \frac{-4}{3}$

$\therefore$ curve is $\frac{-x}{y} = \frac{x^2}{2} - \frac{4}{3}$

It also passes through (3, y)

$\frac{-3}{y} = \frac{9}{2} - \frac{4}{3}$

$\Rightarrow \frac{-3}{y} = \frac{19}{6}$

$\Rightarrow y = -\frac{18}{19}$

11. If $f(x) = \int_1^x \frac{\log_e(t)}{(1+t)} dt$, then $f(e) + f\left(\frac{1}{e}\right)$ is

(1) 0 (2) 1 (3) –1 (4) $\frac{1}{2}$

Ans. (4)

Sol. $f(e) + f\left(\frac{1}{e}\right) = \int_1^e \frac{\ell nt}{1+t} dt + \int_1^{1/e} \frac{\ell nt}{1+t} dt = I_1 + I_2$

$I_2 = \int_1^{1/e} \frac{\ell nt}{1+t} dt$ put $t = \frac{1}{z}$ $dt = -\frac{dz}{z^2}$

$= \int_1^e -\frac{\ell nz}{1+\frac{1}{z}} \times \left(-\frac{dz}{z^2}\right) = \int_1^e \frac{\ell nz}{z(z+1)} dz$

$f(e) + f\left(\frac{1}{e}\right) = \int_1^e \frac{\ell nt}{1+t} dt + \int_1^e \frac{\ell nt}{t(t+1)} dt = \int_1^e \frac{\ell nt}{1+t} + \frac{\ell nt}{t(t+1)} dt$

$= \int_1^e \frac{\ell nt}{t} dt = \int_0^1 u\, du = \left.\frac{u^2}{2}\right|_0^1 = \frac{1}{2}$

12. Consider the system of equation $x + 2y - 3z = a$, $2x + 6y - 11z = b$, $x - 2y + 7z = c$ then

(1) unique solution for $\forall$ a, b, c
(2) infinite solution for $5a = 2b + c$
(3) no solution for all a, b, c
(4) unique solution for $5a = 2b + c$

Ans. (2)

Sol. $D = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 6 & -11 \\ 1 & -2 & 7 \end{vmatrix}$

$= 20 - 2(25) - 3(-10)$
$= 20 - 50 + 30 = 0$

$D_1 = \begin{vmatrix} a & 2 & -3 \\ b & 6 & -11 \\ c & -2 & 7 \end{vmatrix}$

$= 20a - 2(7b + 11c) - 3(-2b - 6c)$
$= 20a - 14b - 22c + 6b + 18c$
$= 20a - 8b - 4c$
$= 4(5a - 2b - c)$

$D_2 = \begin{vmatrix} 1 & a & -3 \\ 2 & b & -11 \\ 1 & c & 7 \end{vmatrix}$

$= 7b + 11c - a(25) - 3(2c - b)$
$= 7b + 11c - 25a - 6c + 3b$
$= -25a + 10b + 5c$
$= -5(5a - 2b - c)$

$D_3 = \begin{vmatrix} 1 & 2 & a \\ 2 & 6 & b \\ 1 & -2 & c \end{vmatrix}$

$= 6c + 2b - 2(2c - b) - 10a$
$= -10a + 4b + 2c$
$= -2(5a - 2b - c)$

for infinite solution

$D = D_1 = D_2 = D_3 = 0$

$\Rightarrow 5a = 2b + c$

13. A function f(k) is defined A to A where A = {1,2,3,4,5...........10}, such that

$$f(k) = \begin{cases} k+1, & k \in \text{odd} \\ k, & k \in \text{even} \end{cases}.$$

If gof(x) = f(x) then number of mapping of g(x) from $A \rightarrow A$ is

(1) ${}^{10}C_5$ (2) 10^5 (3) 5^5 (4) 5!

Ans. (2)

Sol. $g(f(x)) = f(x)$

$\Rightarrow g(x) = x$, when x is even.

$\therefore$ So total number of functions from A to A

$= 10^5 \times 1 = 10^5$

14. If F_1 (A, B, C) = $(\sim A \vee B) \vee (\sim A) \vee (\sim C \wedge (A \vee B))$

F_2 = (A, B, C) = $(A \vee B) \vee (A \rightarrow \sim B)$

Then which of the following is true :

(1) F_1 is Tautology and F_2 is Tautology

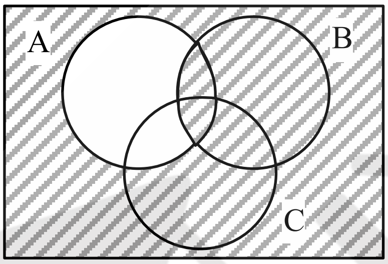
(2) F_1 is Tautology and F_2 is not Tautology

(3) F_1 is not Tautology and F_2 is Tautology

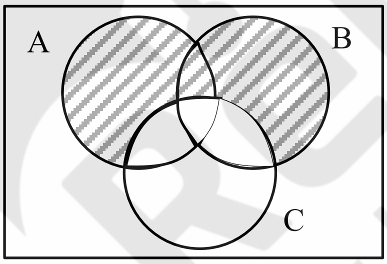
(4) Neither is Tautology

Ans. (3)

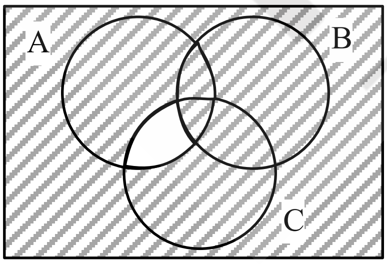
Sol. $(\sim A \vee B) \equiv$



$\sim C \wedge (A \vee B)$

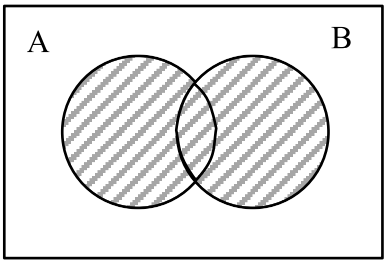


$\therefore$ F_1 :

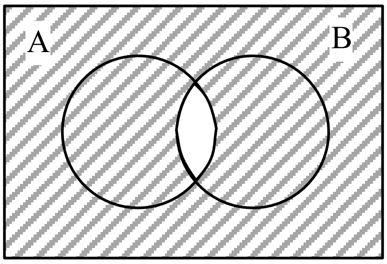


$\Rightarrow$ Not tautology

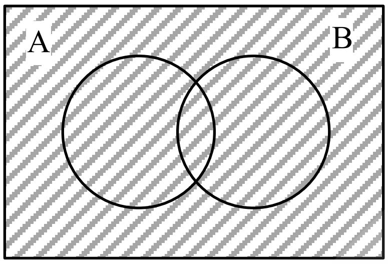
$A \vee B \equiv$



$A \to \sim B$



$\therefore F_2 \to$



Tautology

Truth table for F_1 (A, B, C)

A	B	C	~A	~C	A∨B	~A∨B	~C∧(A∨B)	(~A∨B)∨(~C∧(A∨B))∨~A
T	T	T	F	F	T	T	F	T
T	F	F	F	T	T	F	T	T
T	T	F	F	T	T	T	T	T
T	F	T	F	F	T	F	F	F
F	T	T	T	F	T	T	F	T
F	F	F	T	T	F	T	F	T
F	T	F	T	T	T	T	T	T
F	F	T	T	F	F	T	F	T

Truth table for F_2

A	B	A∨B	~B	A→~B	(A∨B)∨(A→~B)
T	T	T	F	F	T
T	F	T	T	T	T
F	T	T	F	T	T
F	F	F	T	T	T

F_1 not shows tautology and F_2 shows tautology.

15. $\sum_{n=1}^{\infty} \frac{n^2+6n+10}{(2n+1)!} =$

(1) $\frac{41e}{8}+\frac{19e}{8}-10$

(2) $\frac{41e}{8}+\frac{19}{e}-10$

(3) $\frac{41}{8}e-\frac{19}{8e}-10$

(4) $\frac{41}{8}e-\frac{19}{8e}-80$

Ans. (3)

Sol. $\sum_{n=1}^{\infty}\frac{n^2+6n+10}{(2n+1)!}$

put 2n + 1 = r, where r = 3,5,7,.........

$\Rightarrow n = \frac{r-1}{2}$

$$\frac{n^2-6n+10}{(2n+1)!}=\frac{\left(\frac{r-1}{2}\right)^2+3r-3+10}{r!}=\frac{r^2+10r+29}{4r!}$$

Now $\sum_{r=3,5,7}\frac{r(r-1)+11r+29}{4r!}=\frac{1}{4}\sum_{r=3,5,7,........}\left(\frac{1}{(r-2)!}+\frac{11}{(r-1)!}+\frac{29}{r!}\right)$

$$=\frac{1}{4}\left\{\left(\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+........\right)+11\left(\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+........\right)+29\left(\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+........\right)\right\}$$

$$=\frac{1}{4}\left\{\frac{e-\frac{1}{e}}{2}+11\left(\frac{e+\frac{1}{e}-2}{2}\right)+29\left(\frac{e-\frac{1}{e}-2}{2}\right)\right\}$$

$$=\frac{1}{8}\left\{e-\frac{1}{e}+11e+\frac{11}{e}-22+29e-\frac{29}{e}-58\right\}$$

$$=\frac{1}{8}\left\{41e-\frac{19}{e}-80\right\}$$

16. Foot of the perpendicular from the points (3, 4, 1) on the line of intersection of the planes x + 2y + z – 6 = 0 & y + 2z = 4 is

(1) $\left(\frac{10}{7},\frac{12}{7},\frac{8}{7}\right)$ (2) $\left(\frac{10}{7},\frac{-12}{7},\frac{8}{7}\right)$ (3) $\left(\frac{10}{7},\frac{-12}{7},\frac{-8}{7}\right)$ (4) $\left(\frac{-10}{7},\frac{12}{7},\frac{8}{7}\right)$

Ans. (1)

Sol. Let D.R's of line are a, b, c

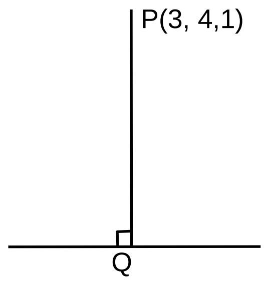
$\therefore$ a + 2b + c = 0

0.a + b + 2c = 0

$\frac{a}{3}=\frac{b}{-2}=\frac{c}{1}$

Points on the line is (–2, 4, 0)

$\therefore$ equation of line is $\frac{x+2}{3}=\frac{y-4}{-2}=\frac{z}{1}=1$



Point Q on the line is $(3\lambda - 2, -2\lambda + 4, \lambda)$

DR's of PQ ; $3\lambda - 5, -2\lambda, \lambda - 1$

DR's of y lines are 3, –2, 1

Since PQ $\perp$ line $\Rightarrow 3(3\lambda - 5) - 2(-2\lambda) + 1(\lambda - 1) = 0$

$\Rightarrow 14\lambda - 16 \Rightarrow \lambda = \frac{8}{7}$

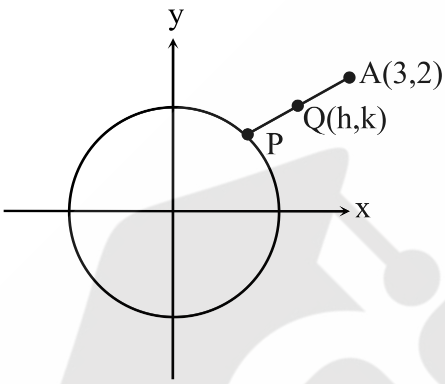
$\therefore\ Q\left(\frac{10}{7}, \frac{12}{7}, \frac{8}{7}\right)$

17. From the point A(3, 2), a line is drawn to any point on the circle $x^2 + y^2 = 1$. if locus of midpoint of this line segment is a circle, the its radius is

(1) $\frac{\sqrt{13}}{2}$ (2) $\frac{1}{2}$ (3) $\frac{\sqrt{11}}{2}$ (4) $\sqrt{11}$

Ans. (1)

Sol.



$\therefore\ P \equiv (2h - 3, 2k - 2) \rightarrow$ on circle

$\therefore\ \left(h - \frac{3}{2}\right)^2 + (k - 1)^2 = \frac{1}{4}$

$\Rightarrow$ radius $= \frac{1}{2}$

18. Given $f(x) = \sin^{-1}x$, $g(x) = \frac{x^2 - x - 2}{2x^2 - x - 6}$, $x \neq 2$ and $g(2) = \lim_{x \to 2} g(x)$ find domain fog(x)

(1) $(-\infty, -2] \cup \left[-\frac{4}{3}, \infty\right)$ (2) $(-\infty, -1] \cup [2, \infty)$

(3) $\left[-2, -\frac{4}{3}\right]$ (4) $(-\infty, 2)$

Ans. (1)

Sol. $g(2) = \lim_{x\to 2} \frac{(x-2)(x+1)}{(2x+3)(x-2)} = \frac{3}{7}$

For domain of fog (x)

$\left|\frac{x^2-x-2}{2x^2-x-6}\right| \le 1 \quad \Rightarrow \quad (x+1)^2 \le (2x+3)^2$

$\Rightarrow (3x+4)(x+2) \ge 0$

$x \in (-\infty, -2] \cup \left(-\frac{4}{3}, \infty\right]$

19. If $f(x) = \begin{cases} 2\sin\left(-\frac{\pi x}{2}\right) & ; \quad x < -1 \\ |ax^2 + x + b| & ; \quad -1 \le x < 1 \\ \sin(\pi x) & ; \quad 1 \le x \end{cases}$ is continuous $\forall$ x $\in$ R, then find (a + b)

(1) –1 (2) 1 (3) 2 (4) –2

Ans. (1)

Sol. If f is continuous at x = –1, then

$f(-1^-) = f(-1)$

$\Rightarrow 2 = |a - 1 + b|$

$\Rightarrow |a + b - 1| = 2$ ..... (i)

similarly

$f(1^-) = f(1)$

$\Rightarrow |a + b + 1| = 0$

$\Rightarrow a + b = -1$

20. If $I_{m,n} = \int_0^1 x^{m-1}.(1-x)^{n-1}\,dx$, m, n ≥ 1 and $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} = \alpha I_{m,n}$, then find 'α'

(1) 1 (2) 2 (3) 0 (4) –1

Ans. (1)

Sol. $I_{m,n} = \int_0^1 x^{m-1}.(1-x)^{n-1}\,dx$ put $x = \frac{1}{y+1}$

$I_{m,n} = \int_\infty^0 \frac{y^{n-1}}{(y+1)^{m+n}}(-1)\,dy = \int_0^\infty \frac{y^{n-1}}{(y+1)^{m+n}}\,dy$ .....(i)

Similarly $I_{m,n} = \int_0^1 x^{n-1}.(1-x)^{m-1}\,dx$

$\Rightarrow I_{m,n} = \int_0^\infty \frac{y^{m-1}}{(y+1)^{m+n}}\,dy$ .....(ii)

From (i) & (ii)

$2I_{m,n} = \int_0^\infty \frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}\,dy$

$\Rightarrow 2I_{m,n} = \int_0^1 \frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}\,dy + \int_1^\infty \frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}\,dy$

Put $y = \frac{1}{z}$

$\Rightarrow 2I_{m,n} = \int_0^1 \frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}\,dy + \int_1^\infty \frac{z^{m-1}+z^{n-1}}{(z+1)^{m+n}}\,dz$

$\Rightarrow I_{m,n} = \int_0^1 \frac{y^{m-1}+y^{n-1}}{(y+1)^{m+1}}\,dy \Rightarrow \alpha = 1$

21. If slope of common tangent to curve $4x^2 + 9y^2 = 36$ and $4x^2 + 4y^2 = 31$ is m then m^2 is equal to

Ans. 3

Sol. E : $\frac{x^2}{9}+\frac{y^2}{4} = 1$ C : $x^2 + y^2 = \frac{31}{4}$

equation of tangent to ellipse

$y = mx \pm \sqrt{9m^2+4}$ ....(i)

equation of tangent to circle

$y = mx \pm \sqrt{\frac{31}{4}m^2+\frac{31}{4}}$ ...(ii)

Comparing equation (i) & (ii)

$9m^2 + 4 = \frac{31m^2}{4}+\frac{31}{4}$

$\Rightarrow 36m^2 + 16 = 31m^2 + 31$

$\Rightarrow 5m^2 = 15$

$\Rightarrow m^2 = 3$

22. If $A=\begin{bmatrix}1&0&0\\0&2&0\\3&0&-1\end{bmatrix}$ and $A^{20}+\alpha A^{19}+\beta A=\begin{bmatrix}1&0&0\\0&4&0\\0&0&1\end{bmatrix}$ then find $(\alpha-\beta)$

Ans. 4

Sol. $A^2=\begin{bmatrix}1&0&0\\0&2&0\\3&0&-1\end{bmatrix}\begin{bmatrix}1&0&0\\0&2&0\\3&0&-1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&4&0\\0&0&1\end{bmatrix}$

$$A^3=\begin{bmatrix}1&0&0\\0&4&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&2&0\\3&0&-1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&8&0\\3&0&-1\end{bmatrix}$$

$$A^4=\begin{bmatrix}1&0&0\\0&4&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&4&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&16&0\\0&0&1\end{bmatrix}$$

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$$A^{19}=\begin{bmatrix}1&0&0\\0&2^{19}&0\\3&0&-1\end{bmatrix}, A^{20}=\begin{bmatrix}1&0&0\\0&2^{20}&0\\0&0&1\end{bmatrix}$$

$$\text{L.H.S} = A^{20}+\alpha A^{19}+\beta A=\begin{bmatrix}1+\alpha+\beta&0&0\\0&2^{20}+\alpha 2^{19}+2\beta&0\\3\alpha+3\beta&0&1-\alpha-\beta\end{bmatrix}$$

$$\text{R.H.S} = \begin{bmatrix}1&0&0\\0&4&0\\0&0&1\end{bmatrix} \Rightarrow \alpha+\beta=0 \text{ and}$$

$$2^{20}+\alpha 2^{19}+2\beta=4$$
$$2^{20}+\alpha(2^{19}-2)=4$$

$$2=\frac{4-2^{20}}{2^{19}-2}=-2$$

$\beta=2\Rightarrow(\alpha-\beta)=4$

23. In normal drawn to the curve at any point passes through a fixed point (a, b). The curve passes through (3, –3) & $(4, -2\sqrt{2})$ such that $a - 2\sqrt{2}\,b = 3$. Find $a^2 + b^2 + ab$

Ans. 9

Sol. Let the equation of normal is $Y - y = -\frac{1}{m}(X - x)$

Satisfy (a, b) in it $b - y = -\frac{1}{m}(a - x)$

$\Rightarrow (b - y)\,dy = (x - a)\,dx$

$by - \frac{y^2}{2} = \frac{x^2}{2} - ax + c$ ……(i)

It passes through (3, –3) & $(4, -2\sqrt{2})$

$\therefore\ -3b - \frac{9}{2} = \frac{9}{2} - 3a + c$

$\Rightarrow -6b - 9 = 9 - 6a + 2c$

$\Rightarrow 6a - 6b - 2c = 18$

$\Rightarrow 3a - 3b - c = 9$ ……(ii)

Also

$-2\sqrt{2}\,b - 4 = 8 - 4a + c$

$4a - 2\sqrt{2}\,b - c = 12$ ……(iii)

Also $a - 2\sqrt{2}\,b = 3$ ……(iv) (given)

(ii) – (iii) $\Rightarrow -a + (2\sqrt{2} - 3)\,b = -3$ ……(v)

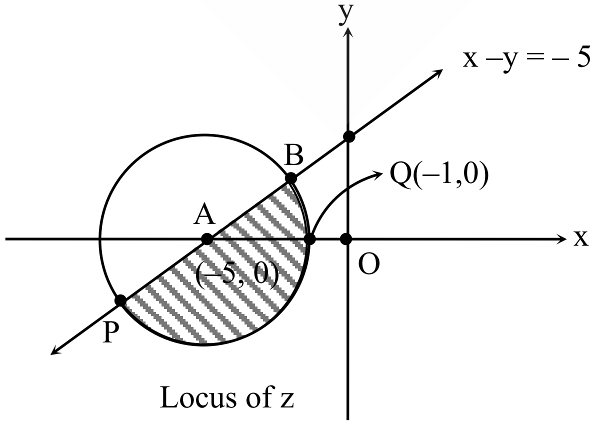
(iv) + (v) $\Rightarrow$ b = 0 a = 3

$\therefore\ a^2 + b^2 + ab = 9$

24. Let $z\,(z \in C)$ satisfy $|z + 5| \le 5$ and $z(1 + i) + \bar{z}(1 - i) \ge -10$ if the maximum value of $|z + 1|^2$ is $\alpha + \beta\sqrt{2}$; then find $\alpha + \beta$

Ans. 48

Sol.



Locus of z

$\therefore\ P \equiv (-5 - 2\sqrt{2}, -2\sqrt{2})$

$\therefore\ (PQ)^2\Big|_{max} = 32 + 16\sqrt{2}$

$\alpha = 32$

$\beta = 16$

$\alpha + \beta = 48$

25. –16, 8, –4, 2 ...... is sequence, whose AM & GM of p^{th} & q^{th} term are the roots of $4x^2 - 9x + 5 = 0$, Find p + q

Ans. 10

Sol. –16, 8, –4, 2 ......

p^{th} term $t_p = -16\left(\frac{-1}{2}\right)^{p-1}$

q^{th} term $t_q = -16\left(\frac{-1}{2}\right)^{q-1}$

Now $\frac{t_p + t_q}{2} = \frac{5}{4}$ & $\sqrt{t_p t_q} = 1$

$\Rightarrow 16^2\left(-\frac{1}{2}\right)^{p+q-2} = 1$

$\Rightarrow 2^8 = (-2)^{(p+q-2)}$

$\Rightarrow p + q = 10$

26. If $x_1, x_2, x_3 \ldots\ldots x_{18}$ are 18 terms and $\sum_{1}^{18}(x_i - \alpha) = 36$, $\sum_{1}^{18}(x_i - \beta)^2 = 90$ variance $(\sigma^2) = 1$ given, then find $|\beta - \alpha|$

Ans. 4

Sol. $\Sigma x_i - 18\alpha = 36$

$\Sigma x_i = 18(\alpha + 2)$ …(i)

$\Sigma x_i^2 + 18\beta^2 - 2\beta\,\Sigma x_i = 90$

$\Sigma x_i^2 + 18\beta^2 - 2\beta \times 18(\alpha + 2) = 90$

$\Sigma x_i^2 = 90 - 18\beta^2 + 36\beta(\alpha + 2)$ …(ii)

$\sigma^2 = 1 \Rightarrow \frac{1}{18}\Sigma x_i^2 - \left(\frac{\Sigma x_i}{18}\right)^2 = 1$

$\Rightarrow \frac{1}{18}(90 - 18\beta^2 + 36\alpha\beta + 72\beta) - \left(\frac{18(\alpha+2)}{18}\right)^2 = 1$

$\Rightarrow 90 - 18\beta^2 + 36\alpha\beta + 72\beta - 18(\alpha + 2)^2 = 18$

$\Rightarrow 5 - \beta^2 + 2\alpha\beta + 4\beta - (\alpha + 2)^2 = 1$

$\Rightarrow 5 - \beta^2 + 2\alpha\beta + 4\beta - \alpha^2 - 4 - 4\alpha = 1$

$- \alpha^2 - \beta^2 + 2\alpha\beta + 4\beta - 4\alpha = 0$

$- \quad (\alpha - \beta)^2 - 4(\alpha - \beta) = 0$

$- (\alpha - \beta)(\alpha - \beta + 4) = 0$

$\Rightarrow \alpha - \beta = -4 \qquad (\alpha \neq \beta)$

$|\beta - \alpha| = 4$

27. It $P_n = \alpha^n + \beta^n$, $\alpha + \beta = 1$, $\alpha\beta = -1$, $P_{n-1} = 11$, $P_{n+1} = 29$ find $(P_n)^2$ (where $n \in N$)

Ans. 324

Sol. Quadratic Equation whose roots are α, β : $x^2 - x - 1 = 0$

$\therefore \alpha^2 = \alpha + 1 \Rightarrow \alpha^n = \alpha^{n-1} + \alpha^{n-2}$

$\beta^2 = \beta + 1 \Rightarrow \beta^n = \beta^{n-1} + \beta^{n-2}$

$\therefore P_n = P_{n-1} + P_{n-2}$

$\Rightarrow P_{n+1} = P_n + P_{n-1}$

$\Rightarrow 29 = P_n + 11 \Rightarrow P_n = 18$

$\Rightarrow (Pn)^2 = 324$

28. The number of four digit numbers whose HCF with 18 is 3 equals

Ans. 1000

Sol. Number must be an odd multiple of 3 and not a multiple of 9

4-digit odd multiples of 3 are

1005, 1011, ……., 9999 $\rightarrow$ 1499

4-digit odd multiples of 9 are

1017, 1035, ……., 9999 $\rightarrow$ 499

$\therefore$ Required numbers $\rightarrow$ 1000

29. Image of a point (1, 0, –1) in the plane $4x - 5y + 2z = 8$ is (α, β, γ). Find $15(\alpha + \beta + \gamma)$

Ans. 4

Sol. $\frac{x-1}{4} = \frac{y-0}{-5} = \frac{z+1}{2} = \frac{-2(-6)}{16+25+4} = \frac{12}{45} = \frac{4}{15}$

$x - 1 = \frac{16}{15} \Rightarrow x = \frac{31}{15}$

$y = -\frac{4}{3}$

$z + 1 = \frac{8}{15} \Rightarrow z = -\frac{7}{15}$

$\alpha = \frac{31}{15}, \beta = -\frac{4}{3}, \gamma = -\frac{7}{15}$

$15(\alpha + \beta + \gamma) = \left(\frac{31}{15} - \frac{4}{3} - \frac{7}{15}\right) \times 15 = 4$

30. If $f(x) = 2x^5 + 5x^4 + 10x^3 + 10x^2 + 10x + 10$ and all real roots of f(x) lie in the interval $(-\alpha, -\alpha+1)$ then 'α' is :

Ans. 2

Sol. $f(-1) = 3 > 0$

$f(-2) = -64 + 80 - 80 + 40 - 20 + 10$

$= -34 < 0$

$\therefore$ At least one root in (–2,–1)

$f'(x) = 10\,(x^4 + 2x^3 + 3x^2 + 2x + 1)$

$$= 10\left(x^2 + \frac{1}{x^2} + 2\left(x + \frac{1}{x}\right) + 3\right)$$

$$= 10\left(\left(x + \frac{1}{x}\right)^2 + 2\left(x + \frac{1}{x}\right) + 1\right)$$

$$= 10\left(\left(x + \frac{1}{x}\right) + 1\right)^2 > 0; \forall x \in R$$

$\therefore$ Exactly one real root in (–2,–1)