JEE Main 2021 August 27 Shift 2 Mathematics

1. The value of $\int_0^1 \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)dx$ is

(A) 0

(B) $\frac{1}{2}$

(C) $\frac{1}{4}$

(D) $\frac{1}{8}$

Ans. (C)

Sol. Given $\mathrm{I} = \int_0^1 \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)dx$

We know that $\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2 = \sin^2\frac{x}{2}+\cos^2\frac{x}{2}+ 2\sin\frac{x}{2}\cos\frac{x}{2} = 1+\sin x$

And $\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2 = \sin^2\frac{x}{2}+\cos^2\frac{x}{2}- 2\sin\frac{x}{2}\cos\frac{x}{2} = 1-\sin x$

$$I = \int_0^1 \cot^{-1}\frac{\left|\sin\frac{x}{2}+\cos\frac{x}{2}\right|+\left|\sin\frac{x}{2}-\cos\frac{x}{2}\right|}{\left|\sin\frac{x}{2}+\cos\frac{x}{2}\right|-\left|\sin\frac{x}{2}-\cos\frac{x}{2}\right|}dx$$

$I = \int_0^1 \cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right)dx$

$$\mathrm{I} = \int_0^1 \cot^{-1}\left(\cot\frac{x}{2}\right)dx$$

$\mathrm{I} = \int_0^1 \frac{x}{2}dx$

$$\mathrm{I} = \left[\frac{x^2}{4}\right]_0^1 = \frac{1}{4}$$

2. If $\sin^4 x+\cos^4 x-\sin x\cos x = 0$ & $x \in [0,\pi]$. Then the value of $\frac{8S}{\pi}$ is (where S is the sum of the solutions of the given equation)

Ans. 2

Sol. Given

$\sin^4 x + \cos^4 x - \sin x \cos x = 0$

We know that $(\sin^2 x + \cos^2 x)^2 = \sin^4 x + \cos^4 x + 2\sin^2 x\cos^2 x$

$\Rightarrow 1 - 2\sin^2 x\cos^2 x - \sin x\cos x = 0$

$\Rightarrow 2\sin^2 x\cos^2 x + \sin x\cos x - 1 = 0$

$\Rightarrow 2\sin^2 x\cos^2 x + 2\sin x\cos x - \sin x\cos x - 1 = 0$

$\Rightarrow (2\sin x\cos x - 1)(\sin x\cos x + 1) = 0$

$\Rightarrow \sin x\cos x = -1, \frac{1}{2}$, ($\sin x\cos x = -1$ rejected)

$\Rightarrow 2\sin x\cos x = 1$

$\Rightarrow \sin 2x = 1$

$\Rightarrow 2x = \dfrac{\pi}{2}$

$\Rightarrow x = \dfrac{\pi}{4}$

$S = \frac{\pi}{4}$

$\Rightarrow \frac{8S}{\pi} = \frac{8\pi}{4\pi} = 2$

3. If $\lim\limits_{x\to\infty}\left(\sqrt{x^2 - x + 1} - ax\right) = b$, then the value of $2(a+b)$ is:

(A) -1

(B) -3

(C) 1

(D) 3

Ans. (C)

Sol. $\lim\limits_{x\to\infty}\left(\sqrt{x^2 - x + 1} - ax\right) = b$

On rationalizing the given expression by multiplying denominator and numerator with $\sqrt{x^2 - x + 1} + ax$

$\Rightarrow \lim\limits_{x\to\infty}\left(\sqrt{x^2 - x + 1} - ax\right) \times \frac{\sqrt{x^2-x+1}+ax}{\sqrt{x^2-x+1}+ax} = b$

$\Rightarrow \lim\limits_{x\to\infty} \frac{(x^2-x+1)-(ax)^2}{\sqrt{x^2-x+1}+ax} = b$

$\Rightarrow$ Limit exist only If $a^2 = 1$

$\therefore a = 1, -1$

$$\Rightarrow \lim_{x\to\infty} \frac{-x+1}{\sqrt{x^2-x+1}+ax} = b$$

Dividing both numerator and denominator by x

$$\Rightarrow \lim_{x\to\infty} \frac{-1+\frac{1}{x}}{\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}+a} = b$$

$$\Rightarrow \frac{-1}{1+a} = b$$

But $a \neq -1$

So, $a = 1$

$$b = \frac{-1}{2}$$

Then $2(a+b) = 1$

4. If the image of a point $P(-1,2,3)$ in the plane $x+y-z-3=0$ is Q and S is a point lying on this plane whose co-ordinates are $(3,2,\beta)$. Then the square of the length of line segment QS is :

(A) 9

(B) 16

(C) 17

(D) $\sqrt{18}$

Ans. (C)

Sol.

Given that Q is the image of a point $P(-1,2,3)$ in plane $x+y-z-3=0$.

Applying formula to find image

$$\frac{x+1}{1} = \frac{y-2}{1} = \frac{z-3}{-1} = \frac{-2(-1+2-3-3)}{1+1+1}$$

$$\Rightarrow \frac{x+1}{1} = \frac{y-2}{1} = \frac{z-3}{-1} = \frac{10}{3}$$

$$\Rightarrow x = \frac{7}{3}, y = \frac{16}{3}, z = \frac{-1}{3}$$

$$\therefore \mathrm{Q} \equiv \left(\frac{7}{3}, \frac{16}{3}, \frac{-1}{3}\right)$$

As S $(3,2,\beta)$ lies on plane, it will satisfy the equation of plane

$$\therefore 3+2-\beta-3 = 0$$

$\beta = 2$

$$\Rightarrow S \equiv (3,2,2)$$

$Q \equiv \left(\frac{7}{3}, \frac{16}{3}, \frac{-1}{3}\right)$ & $S \equiv (3,2,2)$

Using distance formula

$QS = \sqrt{\left(3-\frac{7}{3}\right)^2 + \left(2-\frac{16}{3}\right)^2 + \left(2+\frac{1}{3}\right)^2}$

$QS = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{10}{3}\right)^2 + \left(\frac{7}{3}\right)^2}$

$QS = \sqrt{\frac{4+100+49}{9}} = \sqrt{\frac{153}{9}}$

$QS = \sqrt{17}$

So, $(QS)^2 = 17$

5. Given a curve $P: (y-2)^2 = x-1$. If a tangent is drawn to the curve P at the point whose ordinate is 3 then the area between the tangent, curve and x-axis is:

(A) $\frac{9}{2}$

(B) $\frac{11}{2}$

(C) 9

(D) 11

Ans. (C)

Sol. Given

$P: (y-2)^2 = x-1.$

point whose ordinate is 3

$\Rightarrow y = 3$

$x = 2$

So, point is (2,3)

Curve is $(y-2)^2 = x-1$

Differentiate given equation w.r.t. x, we get

$2(y-2)y' = 1$

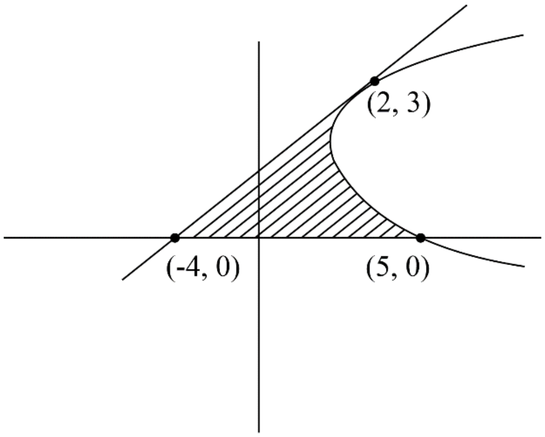
$y' = \frac{1}{2(y-2)}$

$y'|_{(2,3)} = \frac{1}{2}$

So, equation of tangent is

$y-3=\frac{1}{2}(x-2)$

$x-2y+4=0$



Area $=\int_0^3 [(y-2)^2+1-(2y-4)]dy$

Area $=\int_0^3 [y^2-4y+4+1-2y+4]dy$

Area $=\int_0^3 (y^2-6y+9)dy$

Area $=\left(\frac{y^3}{3}-3y^2+9y\right)_0^3=9-27+27 \ = \ 9 \text{ sq. unit}$

6. Perpendicular tangents are drawn from an external point P to the parabola $y^2=16(x-3)$. Then the locus of point P is:

(A) $x=-1$

(B) $x=\frac{1}{2}$

(C) $x=1$

(D) $x=2$

Ans. (A)

Sol.

Given $y^2=16(x-3)$

We know that if perpendicular tangents are drawn from an external point P to the parabola, then point lies on the directrix of parabola.

Locus is directrix of parabola.

$Y^2 = 16X$

Where $X = x - 3, Y = y$.

$4A = 16 \Rightarrow A = 4$

Equation of directrix is

$X + A = 0$

$x - 3 + 4 = 0 \Rightarrow x + 1 = 0$

$\Rightarrow x = -1$

7. $(p \wedge q) \rightarrow ((r \wedge q) \wedge p)$ is a

(A) Tautology

(B) Contradiction

(C) Neither Contradiction nor Tautology

(D) None of these

Ans. (C)

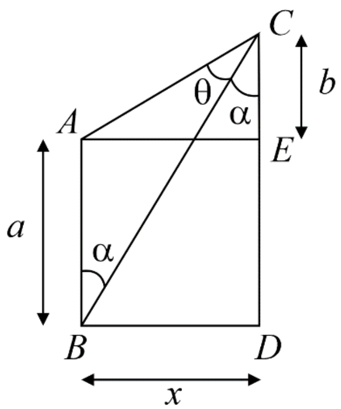
Sol. We need to check $(p \wedge q) \rightarrow ((r \wedge q) \wedge p)$

Let us draw truth table

p	q	r	p ∧ q (I)	r ∧ q	(r ∧ q) ∧ p (II)	I → II
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	T
T	F	F	F	F	F	T
F	T	T	F	T	F	T
F	T	F	F	F	F	T
F	F	T	F	F	F	T
F	F	F	F	F	F	T

As seen in column I → II Its Neither Contradiction nor Tautology.

8. In the given figure, If $\angle ACB = \theta$ and $\tan\theta = \frac{1}{2}$, then the relation between x, a and b is :



(A) $x^2 + 2ax - ab + b^2 = 0$

(B) $x^2 + 2ax + ab + b^2 = 0$

(C) $x^2 - 2ax + ab + b^2 = 0$

(D) $x^2 - 2ax + ab + a^2 = 0$

Ans. (C)

From Diagram

In $\triangle ACE$

$\tan(\theta + \alpha) = \frac{BD}{CE} = \frac{x}{b}$ $\quad (\because BD = AE)$

In $\triangle BCD$

$\tan \alpha = \frac{BD}{DE+EC} = \frac{x}{a+b}$

Given $\tan\theta = \frac{1}{2}$

$\tan(\theta + \alpha) = \frac{\tan\theta + \tan\alpha}{1 - \tan\theta \tan\alpha}$

$\Rightarrow \frac{\frac{1}{2} + \frac{x}{a+b}}{1 - \frac{1}{2} \cdot \frac{x}{a+b}} = \frac{x}{b}$

$\Rightarrow \frac{a+b+2x}{2(a+b)-x} = \frac{x}{b}$

$\Rightarrow ab + b^2 + 2bx = 2ax + 2bx - x^2$

$\Rightarrow x^2 - 2ax + ab + b^2 = 0$

9. Let there be two circles of same radius 5cm touching each other at point $P(1,2)$ and the common tangent of both the circles at P is $4x + 3y = 10$. If centres of two circles are $C_1(\alpha, \beta)$ and $C_2(\gamma, \delta)$, then the value of $|(\alpha + \beta)(\gamma + \delta)|$ is

(A) 20

(B) 40

(C) 25

(D) 35

Ans. (B)

Sol.

Let the point circle at P be $S:\ (x-1)^2+(y-2)^2=0$.

The common tangent of both the circles at P is $L:\ 4x+3y=10$.

Equation of family of circles is $S+\lambda L=0$

$(x-1)^2+(y-2)^2+\lambda(4x+3y-10)=0$

$x^2+y^2+(4\lambda-2)x+(3\lambda-4)y+5-10\lambda=0$

$r=\sqrt{(2\lambda-1)^2+\left(\frac{3}{2}\lambda-2\right)^2-(5-10\lambda)}=5$

$\Rightarrow r^2=4\lambda^2+1-4\lambda+\frac{9}{4}\lambda^2+4-6\lambda-5+10\lambda=25$

$\Rightarrow \frac{25}{4}\lambda^2=25$

$\Rightarrow \lambda^2=4$

$\Rightarrow \lambda=\pm 2$

for $\lambda=2$

$x^2+y^2+6x+2y-15=0; C_1(-3,-1)$

for $\lambda=-2$

$x^2+y^2-10x-10y+25=0; C_2(5,5)$

$C_1(\alpha,\beta)\equiv C_1(-3,-1)$ and $C_2(\gamma,\delta)\equiv C_2(5,5)$

$|(\alpha+\beta)(\gamma+\delta)|=|(-4)(10)|=40$

10. If $(3x^2+4x+3)^2-(K+1)(3x^2+4x+2)(3x^2+4x+3)+K(3x^2+4x+2)^2=0$ has real roots, then the set of all the values of K is

(A) $\left(1,\ \frac{5}{2}\right]$

(B) $\left(\frac{-1}{2},1\right)$

(C) $\left(\frac{1}{2}, 1\right)$

(D) $\left(-1, \frac{5}{2}\right]$

Ans. (A)

Given

$(3x^2+4x+3)^2-(K+1)(3x^2+4x+2)(3x^2+4x+3)+K(3x^2+4x+2)^2=0$

Let $t=(3x^2+4x+2)$

$\Rightarrow (t+1)^2-(K+1)(t)(t+1)+K(t)^2=0$

$\Rightarrow t^2+2t+1-(K+1)(t^2+t)+Kt^2=0$

$\Rightarrow t^2+2t+1-Kt^2-Kt-t^2-t+Kt^2=0$

$\Rightarrow t-\mathrm{K}t+1=0$

$\Rightarrow t(1-\mathrm{K})=-1$

$\Rightarrow t=\frac{1}{K-1}$

$\Rightarrow 3x^2+4x+2-\frac{1}{K-1}=0$

For real roots, $D \geq 0$

$\Rightarrow 16-4\times 3\times\left(2-\frac{1}{K-1}\right)\geq 0$

$\Rightarrow 4 \geq 6-\frac{3}{\mathrm{K}-1}$

$$\Rightarrow \frac{3}{\mathrm{K}-1}\geq 2$$

$\Rightarrow \frac{3-2K+2}{K-1}\geq 0$

$\Rightarrow \frac{(2\mathrm{K}-5)}{(\mathrm{K}-1)}\leq 0$

$K \in \left(1, \frac{5}{2}\right]$

11. Find the remainder when $3\times 7^{22}+2\times 10^{22}-44$ is divided by 18.

(A) 3

(B) 12

(C) 15

(D) 16

Ans. (C)

Sol. Given

$3 \times 7^{22} + 2 \times 10^{22} - 44$

When divided by 18

Writing 7^{22} as $(6+1)^{22}$ and 10^{22} as $(9+1)^{22}$ and using binomial expansion

$3 \times (6+1)^{22} + 2 \times (9+1)^{22} - 44$

$= 3[\,{}^{22}C_0 6^{22} + {}^{22}C_1 6^{21} + \cdots + {}^{22}C_{22}] + 2[\,{}^{22}C_0 9^{22} + {}^{22}C_1 9^{21} + \cdots + {}^{22}C_{22}] - 44$

$= 3[\,{}^{22}C_0 6^{22} + \cdots + {}^{22}C_{21} 6^1] + 3({}^{22}C_{22}) + 2[\,{}^{22}C_0 9^{22} + \cdots + {}^{22}C_{21} 9^1] + 2({}^{22}C_{22}) - 44$

$= 18\lambda + 3 + 18\mu + 2 - 44$

$= 18\delta - 39$

$= 18\delta - 39 - 15 + 15$

$= 18\beta + 15$

12. Find the value of $\int_0^1 \frac{\sqrt{x}dx}{(x+1)(3x+1)(x+3)}$

(A) $\frac{\pi}{4} - \frac{\sqrt{3}\pi}{16}$

(B) $\frac{\pi}{8} + \frac{\sqrt{3}\pi}{16}$

(C) $\frac{\pi}{8} - \frac{\sqrt{3}\pi}{16}$

(D) 0

Ans. (C)

Sol. Given

$$\int_0^1 \frac{\sqrt{x}dx}{(x+1)(3x+1)(x+3)}$$

Let $\sqrt{x} = t$

$= \int_0^1 \frac{2t^2dt}{(t^2+1)(3t^2+1)(t^2+3)}$

$$= \int_0^1 \frac{\left((3t^2+1) - (t^2+1)\right)dt}{(t^2+1)(3t^2+1)(t^2+3)}$$

$= \int_0^1 \left(\frac{1}{(t^2+3)(t^2+1)} - \frac{1}{(t^2+3)(3t^2+1)}\right) dt$

$= \int_0^1 \left(\frac{\mathrm{dt}}{2(\mathrm{t}^2+1)} - \frac{\mathrm{dt}}{2(\mathrm{t}^2+3)} - \frac{1}{8}\frac{3dt}{3\mathrm{t}^2+1} + \frac{\mathrm{dt}}{8(\mathrm{t}^2+3)}\right)$

$= \int_0^1 \left(\frac{dt}{2(t^2+1)} - \frac{1}{8}\frac{3dt}{(3t^2+1)} - \frac{3}{8}\frac{dt}{(t^2+3)}\right)$

$= \left(\frac{1}{2}\tan^{-1} t - \frac{3\sqrt{3}}{8\times 3}\tan^{-1} \sqrt{3}t - \frac{3}{8\sqrt{3}}\tan^{-1} \frac{t}{\sqrt{3}}\right)_0^1$

$= \frac{1}{2}\tan^{-1} 1 - \frac{\sqrt{3}}{8}\tan^{-1} \sqrt{3} - \frac{\sqrt{3}}{8}\tan^{-1} \frac{1}{\sqrt{3}}$

$= \frac{\pi}{8} - \frac{\sqrt{3}}{8} \times \frac{\pi}{3} - \frac{\sqrt{3}}{8} \times \frac{\pi}{6}$

$= \frac{\pi}{8} - \frac{\sqrt{3}\pi}{16}$

13. When $0 < x < 1, y = \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 \cdots\cdots$ Then what is the value of e^{1-y} at $x = \frac{1}{2}$

Ans. 2

Sol.

$y = \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 \cdots\cdots$

$y = \left(1 - \frac{1}{2}\right)x^2 + \left(1 - \frac{1}{3}\right)x^3 + \left(1 - \frac{1}{4}\right)x^4 + \cdots\cdots\cdots$

$= (x^2 + x^3 + x^4 + \cdots\cdots) - \left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\cdots\right)$

Applying sum of infinite GP using $\frac{\mathrm{a}}{1-r}$ and expansion of $\ell n(1-x)$

$= \frac{x^2}{1-x} + x - \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\cdots\right)$

$= \frac{x}{1-x} + \ell n(1-x)$

Put $x = \frac{1}{2} \Rightarrow y = \frac{\frac{1}{2}}{1-\frac{1}{2}} + \ell n(1 - \frac{1}{2}) = 1 - \ln\, 2$

Then, $e^{1-y} = e^{1-1+\ell n2} = e^{\ell n2} = 2$

14. If $\arg(z_1 - z_2) = \frac{\pi}{4}$ and both z_1 & z_2 satisfy $|z-3| = \mathrm{Re}\,(z)$. Then sum of imaginary part of $(z_1 + z_2)$ is

Ans. 6

Let $z_1 = x_1 + iy_1, \mathrm{z}_2 = x_2 + iy_2$

$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

$\because \arg\,(z_1 - z_2) = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\dfrac{y_1-y_2}{x_1-x_2}\right)=\dfrac{\pi}{4}$

$\Rightarrow \left(\dfrac{y_1-y_2}{x_1-x_2}\right)=1$

$\Rightarrow y_1-y_2=x_1-x_2 \cdots\cdots (1)$

As z_1 & z_2 satisfy $|z-3|=\mathrm{Re}\,(z)$

$|z_1-3|=Re(z_1)$

$\Rightarrow (x_1-3)^2+y_1^2=x_1^2\ldots.(2)$

$|z_2-3|=\mathrm{Re}\,(z_2)$

$\Rightarrow (x_2-3)^2+y_2^2=x_2^2\ldots.(3)$

Applying Equation $(2)-(3)$

$\Rightarrow (x_1-3)^2-(x_2-3)^2+(y_1^2-y_2^2)=x_1^2-x_2^2$

$\Rightarrow (x_1-x_2)(x_1+x_2-6)+(y_1-y_2)(y_1+y_2)=(x_1+x_2)(x_1-x_2)$

From equation (1)

$\Rightarrow (x_1-x_2)(x_1+x_2-6)+(x_1-x_2)(y_1+y_2)=(x_1+x_2)(x_1-x_2)$

$\Rightarrow x_1+x_2-6+y_1+y_2=x_1+x_2$

$\Rightarrow y_1+y_2=6$

15. The equation of the plane which passes through the intersection of the planes $\vec{r}\cdot(2\hat{i}+6\hat{j}+\hat{k})=4$ and $\vec{r}\cdot(2\hat{i}+3\hat{j}+6\hat{k})=2$ and parallel to the $y-axis$ is:

(A) $x+9z=0$

(B) $x+11z=0$

(C) $2x+11z=0$

(D) $4x+11z=0$

Ans. (C)

Sol. Given, $\vec{r}\cdot(2\hat{i}+6\hat{j}+\hat{k})=4$ and $\vec{r}\cdot(2\hat{i}+3\hat{j}+6\hat{k})=2$

Equation of plane in Cartesian form $2x+6y+z=4, 2x+3y+6z=2$

Equation of the plane passing through intersection of two planes can be find using family of planes

$P_1+kP_2=0$

$2x + 6y + z - 4 + k(2x + 3y + 6z - 2) = 0$

$\Rightarrow (2 + 2k)x + (6 + 3k)y + (1 + 6k)z = 4 + 2k$

This plane is parallel to $y - axis$.

Direction ratios of $y - axis$ is $(0,1,0)$ and Direction ratios of normal vector to the plane is

$(2 + 2k), (6 + 3k), (1 + 6k)$

$\Rightarrow 0(2 + 2k) + 1(6 + 3k) + 0(1 + 6k) = 0$

$\Rightarrow 3k = -6$

$\Rightarrow k = -2$

$\Rightarrow -2x - 11z = 0$

$\Rightarrow 2x + 11z = 0$