JEE Main 2021 August 27, Shift 1 (Mathematics)

1. The value of $\int_6^{16} \frac{ln(x^2)}{ln(x^2)+ln(x^2-44x+484)} dx$ is equal to

(A) 0

(B) 5

(C) 10

(D) 15

Ans. (B)

Solution:

Let $I = \int_6^{16} \frac{ln(x^2)}{ln(x^2)+ln(x^2-44x+484)} dx$

$I = \int_6^{16} \frac{ln(x^2)}{ln(x^2)+ln(22-x)^2} dx$

$I = \int_6^{16} \frac{2lnx}{2lnx+2ln(22-x)} dx$

$I = \int_6^{16} \frac{lnx}{lnx+ln(22-x)} dx$ ... (i)

using $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$, we get

$$I = \int_6^{16} \frac{ln(22-x)}{ln(22-x)+lnx} \; dx \quad ...(ii)$$

By adding (i) & (ii), we get

$2I = \int_6^{16} dx = [x]_6^{16} = 10$

$I = 5$

2. The value of $\sum_{k=0}^{20} \left({}^{20}C_k\right)^2$ is equal to

(A) ${}^{30}C_{20}$

(B) ${}^{40}C_{20}$

(C) $\left({}^{20}C_{10}\right)^2$

(D) ${}^{40}C_{15}$

Ans. (B)

Sol.

$\sum_{k=0}^{20}\left({}^{20}C_k\right)^2 = \sum_{k=0}^{20} {}^{20}C_k \cdot {}^{20}C_{20-k} = {}^{40}C_{20}$ (which is the coefficient of x^{20} in the expansion of $(1+x)^{40}$)

3. When a biased die is thrown, the probability of occurrence of 1 is $\frac{1}{6}-x$, probability of occurrence of 6 is $\frac{1}{6}+x$ and the probability of occurrence of each remaining face is $\frac{1}{6}$. If two such dice are thrown, then the probability of getting the sum equal to 7 is $\frac{13}{96}$. The value of x is

(A) $\frac{1}{18}$

(B) $\frac{1}{12}$

(C) $\frac{1}{8}$

(D) $\frac{1}{6}$

Ans. (C)

Sol.

Given that

$P_1 = \frac{1}{6}+x, P_6 = \frac{1}{6}-x$

$P_3 = P_4 = P_5 = P_2 = \frac{1}{6}$

Appling the given condition

$P(\,sum = 7) = P(6,1) + P(1,6) + P(5,2) + P(2,5) + P(4,3) + P(3,4)$

$\Rightarrow \left(\frac{1}{6}+x\right)\left(\frac{1}{6}-x\right) + \left(\frac{1}{6}-x\right)\left(\frac{1}{6}+x\right) + \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^2 = \frac{13}{96}$

$\Rightarrow 2\left(\frac{1}{36}-x^2\right) + \frac{4}{36} = \frac{13}{96}$

$\Rightarrow \frac{1}{6} - 2x^2 = \frac{13}{96}$

$\Rightarrow 2x^2 = \frac{1}{6} - \frac{13}{96}$

$\Rightarrow x^2 = \frac{1}{64}$

$\Rightarrow x = \frac{1}{8}, \frac{-1}{8}$

4. Let two points are $A(0,\ 6)$ and $B(2t, 0)$ and M is the mid point of AB. If perpendicular bisector of line joining the points A and B cuts the y-axis at point C, then the locus of mid point of MC is

(A) $x^2 = 3(2-y)$

(B) $2x^2 = 3(3-y)$

(C) $2x^2 = 3(2-y)$

(D) $3x^2 = 2(3-y)$

Ans. (B)

Sol.

$A(0,6)$ and $B(2t,0)$

Since, M is mid point of $AB \Rightarrow M \equiv (t,3)$

Equation of the perpendicular bisector of A and B is

$$y-3=\frac{t}{3}(x-t)$$

y-axis point of intersection will be $C\left(0,3-\frac{t^2}{3}\right)$

Let the mid point of MC is (h,k)

$$\Rightarrow (h,k)\equiv\left(\frac{t}{2},3-\frac{t^2}{6}\right)$$

$2h=t$ and $\frac{t^2}{6}=3-k$

$$\Rightarrow 4h^2=6(3-k)$$

Locus of (h,k) is $2x^2=3(3-y)$

5. If $u=\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)^2\left(1+\frac{3^2}{n^2}\right)^3\ldots\ldots\ldots\left(1+\frac{n^2}{n^2}\right)^n$, then the value of $\lim\limits_{n\to\infty}u^{-4/n^2}$ is :

(A) $\frac{e^2}{4}$

(B) $\frac{e^2}{16}$

(C) $\frac{4}{e}$

(D) $\frac{16}{e^2}$

Ans. (B)

Sol.

$$y=\lim_{n\to\infty}\left(1+\frac{1^2}{n^2}\right)^{\frac{-4}{n^2}}\left(1+\frac{2^2}{n^2}\right)^{\frac{-4}{n^2}\cdot 2}\left(1+\frac{3^2}{n^2}\right)^{\frac{-4}{n^2}\cdot 3}\ldots$$

$$lny=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{-4}{n^2}rln\left(1+\frac{r^2}{n^2}\right)$$

Put $\frac{r}{n}=x$ and $\frac{1}{n}=dx$

$$lny=\int_0^1-4xln(1+x^2)dx$$

Let $1+x^2=t$

$$lny=-2\int_1^2 lntdt$$

$$=-2(tlnt-t)|_1^2$$

$$=-2(2ln2-2+1)$$

$$=-2(2ln2-1)$$

$\ln y=ln\frac{1}{16}+2$

$$y=\frac{1}{16}e^2$$

6. If$(sin^{-1}x)^2-(cos^{-1}x)^2=a$, then the value of $(2x^2-1)$ is:

(A) $sin\left(\frac{2a}{\pi}\right)$

(B) $cos\left(\frac{2a}{\pi}\right)$

(C) $sin\left(\frac{4a}{\pi}\right)$

(D) $cos\left(\frac{4a}{\pi}\right)$

Ans. (A)

Sol.

$$(sin^{-1}x)^2-(cos^{-1}x)^2=a,$$

$$\Rightarrow (sin^{-1}x+cos^{-1}x)(sin^{-1}x-cos^{-1}x)=a$$

$\Rightarrow \frac{\pi}{2}(sin^{-1}x-cos^{-1}x)=a$ (using $sin^{-1}x+cos^{-1}x=\frac{\pi}{2}$)

$$\Rightarrow \frac{\pi}{2}-2cos^{-1}x=\frac{2a}{\pi}$$

$$\Rightarrow 2cos^{-1}x=\frac{\pi}{2}-\frac{2a}{\pi}$$

$$\Rightarrow cos^{-1}(2x^2-1)=\left(\frac{\pi}{2}-\frac{2a}{\pi}\right)$$

$$\Rightarrow 2x^2-1=cos\left(\frac{\pi}{2}-\frac{2a}{\pi}\right)$$

$$\Rightarrow 2x^2-1=sin\left(\frac{2a}{\pi}\right)$$

7. If $k=\frac{3}{2}x^2+\frac{5}{3}x^3+\frac{7}{4}x^4+\cdots\infty$ then the value of k is

(A) $\frac{x^2+x}{1-x}+ln(1-x)$

(B) $\frac{x^2+x}{1+x}+ln(1+x)$

(C) $1+x+ln(1-x)$

(D) 0

Ans. (A)

Sol.

Here, $k=\left(2-\frac{1}{2}\right)x^2+\left(2-\frac{1}{3}\right)x^3+\left(2-\frac{1}{4}\right)x^4+\cdots$ ...

$$k=2[x^2+x^3+x^4+\cdots\ldots]-\left(\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots.\right)$$

$k = \frac{2x^2}{1-x} + x - \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots ..\right)$ (using formula of G.P.)

$k = \frac{2x^2+x-x^2}{1-x} + ln(1-x)$ (using expansion of $ln(1-x)$)

$k = \frac{x^2+x}{1-x} + ln(1-x)$

8. Consider an ellipse $E: \frac{x^2}{b^2} + \frac{y^2}{4a^2} = 1$. If the minimum area enclosed between the tangent of the ellipse and the coordinate axis is kab, then the value of k is

(A) 1

(B) 2

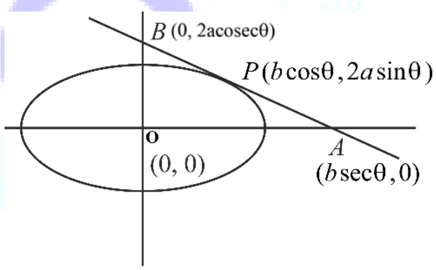
(C) 3

(D) 4

Ans. (A)

Sol.

The equation of tangent at $P(bcos\theta, 2asin\theta)$ is



$$\frac{xcos\theta}{b} + \frac{ysin\theta}{2a} = 1$$

$$A(bsec\theta, 0), B(0, 2acosec\theta)$$

Area of triangle $OAB = \frac{1}{2} \times bsec\theta \times 2acosec\theta$

$$= \frac{ab}{cos\theta sin\theta} = \frac{2ab}{sin2\theta}$$

$$\Delta = \frac{2ab}{sin2\theta}$$

$\Delta_{min} = 2ab$ (as the value of $sin2\theta$ is maximum at 1)

So $k = 2$

9. If $\frac{2+4+6+8+\cdots......+2y}{3+6+9+12+\cdots......+3y} = \frac{4}{log_{10}x}$, then the value of $y = log_{10}x + log_{10}x^{\frac{1}{3}} + log_{10}x^{\frac{1}{9}} + \cdots$ is

(A) 4

(B) 6

(C) 9

(D) 12

Ans. (C)

Sol. $\frac{2(1+2+3+4+\cdots\ldots\ldots+y)}{3(1+2+3+4+\cdots\ldots\ldots+y)}=\frac{4}{log_{10}x}$ (taking 2 common from numerator and 3 common from denominator)

$\Rightarrow log_{10}x=6 \;\; \Rightarrow \;\; x=10^6\ldots(1)$

So, $y=log_{10}x+log_{10}x^{\frac{1}{3}}+log_{10}x^{\frac{1}{9}}+\cdots$

$=log_{10}x+\frac{1}{3}log_{10}x+\frac{1}{9}log_{10}x+\cdots$

$=\left(1+\frac{1}{3}+\frac{1}{9}+\cdots\right)log_{10}x=\frac{1}{1-\frac{1}{3}}\times 6=9$

10. If α and β are the roots of $x^2+bx+c=0$, then the value of $\lim_{x\to\beta}\frac{e^{2(x^2+bx+c)}-1-2(x^2+bx+c)}{(x-\beta)^2}$ is

(A) $4(b^2-4c)$

(B) $2(b^2-2c)$

(C) $2(b^2-4c)$

(D) $2(b^2+4c)$

Ans. (C)

Sol. We know difference of roots, $|\beta-\alpha|=\sqrt{b^2-4c}$

Here, $\lim_{x\to\beta}\frac{e^{2(x-\alpha)(x-\beta)}-1-2(x-\alpha)(x-\beta)}{(x-\beta)^2}$

Let $x-\beta=h$

$\lim_{h\to 0}\frac{e^{2(\beta-\alpha+h)h}-1-2(\beta-\alpha+h)h}{h^2}$

$$=\lim_{h\to 0}\frac{(1+2(\beta-\alpha+h)h+\frac{(2(\beta-\alpha+h)h)^2}{2!}\ldots)-1-2h(\beta-\alpha+h)}{h^2}=\lim_{h\to 0}\frac{\frac{(2(\beta-\alpha+h)h)^2}{2!}+\cdots}{h^2}$$

$$=\;2(\beta-\alpha)^2=2(b^2-4c)$$

11. A 20 m length wire is cut into 2 pieces. One piece is used to make a square and the other piece is used to make a regular hexagon. The length of the side of the hexagon, so that sum of areas of square and hexagon is minimum, will be:

(A) $\frac{40\sqrt{3}}{3-2\sqrt{3}}$

(B) $\frac{10}{3+2\sqrt{3}}$

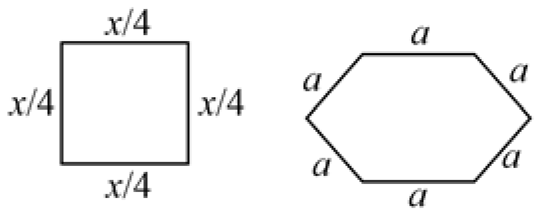
(C) $\frac{10}{3-2\sqrt{3}}$

(D) $\frac{40\sqrt{3}}{3+2\sqrt{3}}$

Ans. (B)

Sol.

Let the length of 2 pieces be x and $20-x$



$$6a = 20 - x$$
$$a = \frac{20 - x}{6}$$

Area of square $A' = \frac{x^2}{16}$

Area of hexagon $A" = 6 \times \frac{\sqrt{3}}{4} a^2$

$$A" = \frac{3\sqrt{3}}{2} \cdot \left(\frac{20-x}{6}\right)^2$$

Sum of both area, $A = A' + A"$

$$A = \frac{x^2}{16} + \frac{\sqrt{3}}{24}(20-x)^2$$

$$\frac{dA}{dx} = \frac{x}{8} - \frac{\sqrt{3}}{12}(20-x) = \frac{3x - 40\sqrt{3} + 2\sqrt{3}x}{24}$$

$$\frac{dA}{dx} = 0 \Rightarrow x = \frac{40\sqrt{3}}{3+2\sqrt{3}} = \frac{40}{\sqrt{3}+2} = 40(2-\sqrt{3})$$

$$\frac{d^2A}{dx^2} = \frac{3+2\sqrt{3}}{24} > 0$$

So area will be minimum when $x = 40(2-\sqrt{3})$

So length of side of hexagon $= \frac{20-40(2-\sqrt{3})}{6}$

$$= \frac{10-40+20\sqrt{3}}{3} = \frac{(20\sqrt{3}-30)}{3} = \frac{10(2\sqrt{3}-3)}{3} = \frac{10}{2\sqrt{3}+3}$$

12. If point (x, y) satisfy the relation $x^2 + 4y^2 - 4x + 3 = 0$, then

(A) $x \in [1,3];\ y \in \left[-\frac{1}{3}, \frac{1}{3}\right]$

(B) $x \in [1,3];\ y \in [1,3]$

(C) $x \in [1,3];\ y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

(D) $x \in [-2, 2];\ y \in [-1, 1]$

Ans. (3)

Sol. $(x-2)^2 + 4y^2 = 1$

$$(x-2)^2 + \frac{y^2}{\frac{1}{4}} = 1$$

Which is the equation of ellipse with end points of major axis at $(\pm 1, 0)$ and end points of minor axis at $(0, \pm\frac{1}{2})$

Hence $-1 \leq x - 2 \leq 1$ and $-\frac{1}{2} \leq y \leq \frac{1}{2}$

$1 \leq x \leq 3$ and $-\frac{1}{2} \leq y \leq \frac{1}{2}$

13. Let $1, 2, 3, \ldots \ldots, n$ are n variates having variance 14, then the value of n is equal to

Ans. (13)

Sol. We know tha,

variance $= \frac{1^2+2^2+3^2+\cdots..+n^2}{n} - \left(\frac{(1+2+3+\cdots..+n)}{n}\right)^2$

$$\Rightarrow 14 = \frac{n(n+1)(2n+1)}{6n} - \left(\frac{n+1}{2}\right)^2$$

$$\Rightarrow 14 = \frac{n+1}{12}[2(2n+1) - 3(n+1)]$$

$$\Rightarrow n^2 = 169$$

$$\Rightarrow n = 13$$

14. If $\frac{z-i}{z+2i} \in R$ and $z \in C$, then the correct statement is:

(1) z is a straight line in argand plane

(2) z has only 1 value

(3) z has only 2 values

(4) z is a circle in argand plane

Ans. (1)

Sol. $\left(\frac{z-i}{z+2i}\right) = \overline{\left(\frac{z-i}{z+2i}\right)}$

$$\left(\frac{z-i}{z+2i}\right) = \left(\frac{\bar{z}+i}{\bar{z}-2i}\right)$$

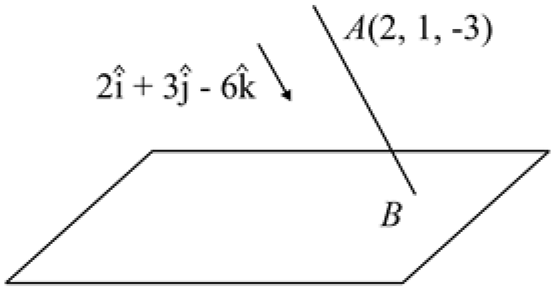
Solving we get

$z = -\bar{z}$ which is the locus of y-axis

15. The distance of the point $(2, 1, -3)$ parallel to the vector $(2\hat{i} + 3\hat{j} - 6\hat{k})$ from the plane $2x + y + z + 8 = 0$ is:

Ans. 70

Sol.



Equation of line AB

$\frac{x-2}{2} = \frac{y-1}{3} = \frac{z+3}{-6} = \lambda$

Let $B\ (2\lambda + 2, 3\lambda + 1, -6\lambda - 3)$

Since B satisfy the given plane $2x + y + z + 8 = 0$

$$2(2\lambda + 2) + (3\lambda + 1) - 6\lambda - 3 + 8 = 0$$

$$\lambda = -10$$

$$B(-18, -29, 57)$$

Distance $AB = \sqrt{20^2 + 30^2 + 60^2} = \sqrt{400 + 900 + 3600} = 70$

16. If $\int \frac{1}{(x^2+x+1)^2} dx = a\,tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + b\left(\frac{2x+1}{x^2+x+1}\right) + C$, then value of $9\left(\frac{a}{\sqrt{3}} + b\right)$ is equal to

Ans. 7

Sol. Given $\int \frac{1}{(x^2+x+1)^2} dx = a\,tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + b\left(\frac{2x+1}{x^2+x+1}\right) + C$

Let $I = \int \frac{1}{(x^2+x+1)^2} dx$

Converting quadratic in denominator into a perfect square

$$I = \int \frac{1}{\left[\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right]^2} dx$$

Substitute $x + \frac{1}{2} = \frac{\sqrt{3}}{2} tan\theta$

$$dx = \frac{\sqrt{3}}{2} sec^2\theta d\theta$$

$$\Rightarrow \mathrm{I} = \int \frac{\frac{\sqrt{3}}{2} sec^2\theta}{\left(\frac{3}{4}\tan^2\theta + \frac{3}{4}\right)^2} d\theta$$

$$= \int \frac{\frac{\sqrt{3}}{2} sec^2\theta}{\left(\frac{3}{4} sec^2\theta\right)^2} d\theta$$

$$= \frac{8}{3\sqrt{3}} \int cos^2\theta d\theta$$

$$= \frac{8}{3\sqrt{3}} \int \frac{1 + cos2\theta}{2} d\theta$$

$$= \frac{4}{3\sqrt{3}} \left[\theta + \frac{sin2\theta}{2}\right] + C$$

As we know $x + \frac{1}{2} = \frac{\sqrt{3}}{2} tan\theta$

$\theta = tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$ and $sin2\theta = \frac{2tan\theta}{1+tan^2\theta}$

$$= \frac{4}{3\sqrt{3}} tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{4}{3\sqrt{3}} \cdot \frac{2\left(\frac{2x+1}{\sqrt{3}}\right)}{2\left[1 + \left(\frac{2x+1}{\sqrt{3}}\right)^2\right]} + C$$

$$= \frac{4}{3\sqrt{3}} tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{4}{3\sqrt{3}} \cdot \frac{\frac{2x+1}{\sqrt{3}}}{1 + \left(\frac{2x+1}{\sqrt{3}}\right)^2} + C$$

$$= \frac{4}{3\sqrt{3}} tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{1}{3} \cdot \frac{2x+1}{(x^2 + x + 1)} + C$$

So, $a = \frac{4}{3\sqrt{3}}, b = \frac{1}{3}$

Now, $9\left(\frac{a}{\sqrt{3}} + b\right) = 9\left(\frac{4}{9} + \frac{1}{3}\right) = 7$

17. $\{(p \wedge (p \to q)) \wedge (q \to r)\} \to r$ is equal to ;

(A) $q \to \sim r$

(B) $p \to \sim r$

(C) fallacy

(D) tautology

Ans. (4)

Sol. Given

$\{(p \wedge (p \to q)) \wedge (q \to r)\} \to r$ $(Applying\ p \to q = \sim p \vee q)$

$\equiv \{(p \wedge (\sim p \vee q)) \wedge (q \rightarrow r)\} \rightarrow r$ (*Applying distributive property*)

$\equiv \{((p \wedge \sim p) \vee (p \wedge q)) \wedge (q \rightarrow r)\} \rightarrow r$ (*Applying* $p \wedge \sim p = f$)

$\equiv \{(f \vee (p \wedge q) \wedge (q \rightarrow r)\} \rightarrow r$ (*Applying* $f \vee p = p$ & $p \rightarrow q = \sim p \vee q$)

$\equiv \{(p \wedge q) \wedge (\sim q \vee r)\} \rightarrow r$ (*Applying distributive property*)

$\equiv \{((p \wedge q) \wedge \sim q) \vee ((p \wedge q) \wedge r)\} \rightarrow r$ (*Applying* $(p \wedge q) \wedge \sim q = f$)

$\equiv \{(f \vee ((p \wedge q) \wedge r)\} \rightarrow r$ (*Applying* $f \vee p = p$)

$\equiv \{(p \wedge q) \wedge r\} \rightarrow r$ (*Applying* $p \rightarrow q = \sim p \vee q$)

$\equiv \sim((p \wedge q) \wedge r) \vee r$

$\equiv (\sim(p \wedge q) \vee \sim r) \vee r \equiv t$

which is a tautology.

18. In a ΔABC if $\frac{sinA}{sinB} = \frac{sin(C-A)}{sin(B-C)}$, then

(A) b^2, a^2, c^2 are in $A.P.$
(B) a^2, c^2, b^2 are in $A.P.$
(C) $a^2 - b^2, b^2 - c^2, c^2 - a^2$ are in $G.P.$
(D) a^2, b^2, c^2 are in $G.P.$
Ans. (2)
Sol. Given

$\frac{sinA}{sinB} = \frac{sin(C-A)}{sin(B-C)}$

$A + B + C = \pi$

$$\Rightarrow \frac{sin(\pi - (B + C))}{sin(\pi - (A + C))} = \frac{sin(C - A)}{sin(B - C)}$$

$$\Rightarrow \frac{sin(B + C)}{sin(A + C)} = \frac{sin(C - A)}{sin(B - C)}$$

$\Rightarrow sin(B + C)sin(B - C) = sin(C - A)sin(A + C)$

Using the formula $sin(A + B)sin(A - B) = sin^2 A - sin^2 B$

$\Rightarrow sin^2 B - sin^2 C = sin^2 C - sin^2 A$

$\Rightarrow sin^2 A, sin^2 C, sin^2 B$ are in $A.P.$

$\Rightarrow a^2, c^2, b^2$ are in $A.P.$ (using sine rule)

19. The equation of curve passing through the point $(2, -2)$ and satisfying the differential equation $y + x\frac{dy}{dx} = x^2$ is given by:

(A) $x^3 - 3xy = -20$
(B) $x^3 + 3xy = 20$
(C) $3xy - x^3 = 10$
(D) $x^3 - 3xy = 20$
Ans. (4)

Sol. $y + x\frac{dy}{dx} = x^2$

$\frac{dy}{dx} + \frac{1}{x}y = x$

$\text{IF} = e^{\int \frac{1}{x}dx} = e^{lnx} = x$

Therefore, the general solution of the differential equation will be

$yx = \int x^2 dx$

$xy = \frac{x^3}{3} + c$

This curve passes through the point $(2, -2)$, so we get

$-4 = \frac{8}{3} + c$

$c = -\frac{20}{3}$

$\therefore\ xy = \frac{x^3}{3} - \frac{20}{3}$

So $3xy = x^3 - 20$

Hence, the equation of curve is $x^3 - 3xy = 20$

20. Given matrix $A = \begin{bmatrix} 0 & 2 \\ x & -1 \end{bmatrix}$. If $A(A^3 + 3I) = 2I$ (where I be a unit matrix), then the possible value of x is:

(A) $-\frac{1}{2}$

(B) $\frac{1}{2}$

(C) 1

(D) 2

Ans. (2)

Sol. Given $A^4 + 3AI = 2I$

$A^4 = 2I - 3A$ ...(i)

The characteristic equation is $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 0 - \lambda & 2 \\ x & -1 - \lambda \end{vmatrix} = 0$

$\Rightarrow \lambda + \lambda^2 - 2x = 0$

So, $A^2 + A - 2xI = 0$

$A^2 = 2xI - A$

$A^4 = A^2 + 4x^2 \mid -2(2xA)$

$A^4 = 2xI - A + 4x^2 I - 4xA$

$= (2x + 4x^2)I - A(1 + 4x)$

On comparing with equation (i), we get,

$4x + 1 = 3$

$4x = 2$

$x = \frac{1}{2}$

21. $\vec{a}$ and $\vec{b}$ are two perpendicular vectors with $\vec{a} = \hat{\imath} + 5\hat{\jmath} + \alpha\hat{k}$, $\vec{b} = \hat{\imath} + 3\hat{\jmath} + \beta\hat{k}$. Also, $\vec{c} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$ and $|\vec{b} \times \vec{c}| = 5\sqrt{3}$. The maximum value of $|\vec{a}|^2$ is equal to

Ans. 90

Sol. $\vec{a}$ and $\vec{b}$ are perpendicular, so

$\vec{a} \cdot \vec{b} = 0$

$\Rightarrow 1 + 15 + \alpha\beta = 0$

$\Rightarrow \alpha\beta = -16 \;\; \dots(1)$

$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & \beta \\ 1 & -2 & 3 \end{vmatrix}$

$\Rightarrow \vec{b} \times \vec{c} = \hat{\imath}(9 + 2\beta) - \hat{\jmath}(3 - \beta) + \hat{k}(-2 - 3)$

Given, $|\vec{b} \times \vec{c}| = 5\sqrt{3}$

$\Rightarrow (9 + 2\beta)^2 + (3 - \beta)^2 + 25 = 75$

$\Rightarrow 5\beta^2 + 30\beta + 40 = 0$

$\Rightarrow \beta = -2, -4$

$\beta = -2 \Rightarrow \alpha = 8$ and $\beta = -4 \Rightarrow \alpha = 4$

For the maximum value of $|\vec{a}|^2, \alpha = 8$

So, the maximum value of $|\vec{a}|^2$ is $1 + 25 + 64 = 90$

22. A tangent and normal are drawn at the point $P(2, -4)$ of parabola $y^2 = 8x$ which cut the directrix at points A and B respectively. A given point $Q(h, k)$ is used to form a square $AQBP$. Then the absolute value of $h + k$ is equal to

Ans. 10

The equation of tangent at $(2, -4)$ to the parabola is

$T = 0 \Rightarrow -4y = 4(x + 2)$

$\Rightarrow x + y + 2 = 0 \;\; \dots(1)$

Normal to the parabola at $(2, -4)$ is

$x - y = k$

$\Rightarrow x - y = 6 \;\; \dots(2)$

Directrix of parabola is $x = -2 \;\; \dots(3)$

$\Rightarrow A \equiv (-2,0)$ and $B \equiv (-2, -8)$

As, $AQBP$ is a square. So, the mid-point of AB will be same as the mid-point of PQ

$\Rightarrow (-2, -4) = \left(\frac{h+2}{2}, \frac{k-4}{2}\right)$

$\Rightarrow h = -6, \; k = -4$

$\Rightarrow h + k = -10$

23. If there are three sets defined on set of real numbers $P = \{x: |x - 2| > 1\}, Q = \left\{x: \sqrt{x^2 - 3} > 1\right\}$ and $R = \{x: |x - 4| > 2\}$, then the number of integer elements in the set $(P \cap Q \cap R)'$ is

Ans. 9

Sol.

Set $P = x \in (-\infty, 1) \cup (3, \infty)$

Set $Q = x \in (-\infty, -2) \cup (2, \infty)$

Set $R = x \in (-\infty, 2) \cup (6, \infty)$

$P \cap Q \cap R = (-\infty, -2) \cup (6, \infty)$

$(P \cap Q \cap R)' = [-2,6]$

Number of integral values $= 9$

24. Number of five-digit numbers divisible by 55, formed using the digits 1, 2, 3, 4, 5 & 6 (repetition not allowed) is

Ans. 12

Sol.

A five-digit number $abcde$ is divisible by 55 if $e = 5$ and $abcd5$ is divisible by 11.

So, $|(a + c + 5) - (b + d)|$ should also be divisible by 11 as per the rule of divisibility of 11

When $(a, c) \equiv (6,3)$ and $(b, d) \equiv (1,2) \Rightarrow 4\ ways$

When $(a, c) \equiv (6,4)$ and $(b, d) \equiv (1,3) \Rightarrow 4\ ways$

When $(a, c) \equiv (2,3)$ and $(b, d) \equiv (6,4) \Rightarrow 4\ ways$

Hence, total cases $= 12$

25. If $y^{\frac{1}{4}} + y^{-\frac{1}{4}} = 2x$ is a curve satisfying $\frac{d^2y}{dx^2}(x^2 - 1) + bx\frac{dy}{dx} + ay = 0$, then the absolute value of $a + b$ is equal to

Ans. 15

Solution:

Differentiating $y^{\frac{1}{4}} + y^{-\frac{1}{4}} = 2x$, we get

$\frac{1}{4}\left(y^{\frac{-3}{4}} - y^{-\frac{5}{4}}\right)\frac{dy}{dx} = 2$

$\Rightarrow \left(y^{\frac{1}{4}} - y^{\frac{-1}{4}}\right)\frac{dy}{dx} = 8y$

$\Rightarrow \sqrt{\left(y^{\frac{1}{4}} + y^{\frac{-1}{4}}\right)^2 - 4} = 8y\frac{dx}{dy}$

$\Rightarrow \sqrt{4x^2 - 4} = 8y\frac{dx}{dy}$

$\Rightarrow 2\sqrt{x^2 - 1}\frac{dy}{dx} = 8y$

Now, differentiating w.r.t. x, we get

$\sqrt{x^2-1}\frac{d^2y}{dx^2}+\frac{x}{\sqrt{x^2-1}}\frac{dy}{dx}=4\frac{dy}{dx}$

$\Rightarrow (x^2-1)\frac{d^2y}{dx^2}+x\cdot\frac{dy}{dx}=4\sqrt{x^2-1}\frac{dy}{dx}$

$\Rightarrow (x^2-1)\frac{d^2y}{dx^2}+x\cdot\frac{dy}{dx}=16y$

Now, comparing with $\frac{d^2y}{dx^2}(x^2-1)+bx\frac{dy}{dx}+ay=0$, we get

$a=-16, b=1$

26. If $\frac{dy}{dx}=\frac{-\sin x}{2+\cos x}(y+3), y(0)=1$, then $y\left(\frac{\pi}{3}\right)$ is

(A) $\frac{4}{3}$

(B) $\frac{7}{3}$

(C) $\frac{2}{3}$

(D) 3

Ans: (B)

Solution:

$$\frac{dy}{y+3}=\frac{-\sin x}{2+\cos x}dx$$

$\Rightarrow \ln(y+3)=\ln(2+\cos x)+\ln C$

$\Rightarrow y+3=C(2+\cos x)$

At $x=0, y=1$

$4=3C\Rightarrow C=\frac{4}{3}$

$y=\frac{4}{3}(2+\cos x)-3$

Hence $y\left(\frac{\pi}{3}\right)=\frac{4}{3}\left(2+\frac{1}{2}\right)-3=\frac{2}{3}\cdot 5-1=\frac{7}{3}$