JEE Main 2021 August 26, Shift 1 (Mathematics)

1. Sum of an infinite G. P. with first term a and common ratio r is 15 and the sum of the squares of the terms of the same G. P. is 150 . The sum of an infinite G. P. with first term a and common ratio r^2 is:

(A) $\frac{25}{2}$

(B) $\frac{1}{5}$

(C) $\frac{27}{5}$

(D) $\frac{23}{5}$

Answer: (A)

Solution: Give, that

$a+ar+ar^2 \ldots .=15$

$\frac{a}{1-r}=15 \ldots$ (i)

$a^2+a^2r^2+a^2r^4 \ldots .=150$

$\frac{a^2}{1-r^2}=150 \ldots$ (ii)

From equation (i) and (ii) $\frac{(1-r)^2}{1-r^2}=\frac{150}{225}$

$\Rightarrow\ 3+3r^2-6r=2-2r^2$

$\Rightarrow\ 5r^2-6r+1=0$

$\Rightarrow r=1, \frac{1}{5}$

Now, $\Rightarrow r=\frac{1}{5}$ and $a=12, r\neq 1$ as it is infinite G. P.

$a+ar^2+ar^4 \ldots \ldots=\frac{a}{1-r^2}$

$=\frac{12}{1-\frac{1}{25}}=\frac{25}{2}$

2. If $(1+x)^{20}=\sum_{r=0}^{20}{}^{20}C_r x^r$, then the value of $\sum_{r=0}^{20} r^2\ {}^{20}C_r$ is:

(A) $2^{20}(105)$

(B) $2^{18}(105)$

(C) $2^{19}(205)$

(D) $2^{20}(95)$

Answer: (A)

Solution:

$$(1+x)^{20}=\sum_{r=0}^{20} {}^{20}C_r\, x^r$$

Now $\sum_{r=0}^{20} r^2\ {}^{20}C_r$

$$=\sum_{r=0}^{20} r(r-1+1)\ {}^{20}C_r$$

$$=\sum_{r=0}^{20}(r)(r-1)\ {}^{20}C_r+\sum_{r=0}^{20} r\cdot {}^{20}C_r$$

$=20\times 19\times \sum_{r=0}^{20} {}^{18}C_{r-2}+20\times \sum_{r=0}^{20} {}^{19}C_{r-1}$ (using ${}^{n}C_r=\frac{n}{r}\,{}^{n-1}C_{r-1}$)

$=380\times 2^{18}+20\times 2^{19}$

$=420\times 2^{18}$

$=(105)2^{20}$

3. If probability of independent events A and B are $P(A)=p, P(B)=2p$ and probability of exactly one of A and B is $\frac{5}{9}$, then the value of p is

(A) $\frac{2}{5}$

(B) $\frac{1}{3}$

(C) $\frac{7}{12}$

(D) $\frac{1}{12}$

Answer: (B)

Solution:

$P(A)=p, \qquad P(B)=2p$

$P($ exactly one$)=\frac{5}{9}$

$$\Rightarrow P(A)+P(B)-2P(A\cap B)=\frac{5}{9}$$

$\Rightarrow\ p+2p-2\cdot p\cdot 2p=\frac{5}{9}$ $\{A$ and B are independent$\}$

$\Rightarrow 27p-36p^2=5$

$$\Rightarrow 36p^2-27p+5=0\Rightarrow p=\frac{5}{12},\frac{1}{3}$$

4. The value of integration $\int_{-1/2}^{1/2}\left[\left(\frac{x-1}{x+1}\right)^2+\left(\frac{x+1}{x-1}\right)^2-2\right]^{1/2} dx$ is:

(A) $4\ell n\left(\frac{3}{2}\right)$

(B) $4\ell n\left(\frac{3}{4}\right)$

(C) $4\ell n\left(\frac{4}{3}\right)$

(D) $2\ell n\left(\frac{3}{2}\right)$

Answer: (C)

Solution:

$$\int_{-1/2}^{1/2}\left[\left(\frac{x-1}{x+1}\right)^2+\left(\frac{x+1}{x-1}\right)^2-2\right]^{1/2}dx$$

$$\Rightarrow \int_{-1/2}^{1/2}\left|\frac{x-1}{x+1}-\frac{x+1}{x-1}\right|dx$$

$$=\int_{-1/2}^{1/2}\left|\frac{(x-1)^2-(x+1)^2}{x^2-1}\right|dx$$

$$=\int_{-1/2}^{1/2}\left|\frac{-4x}{x^2-1}\right|dx$$

$$=8\int_0^{1/2}\frac{|x|}{|x^2-1|}dx\ \left\{\text{as } \frac{|x|}{|x^2-1|} \text{ is an even function}\right\}$$

$$=4\int_0^{1/2}\frac{2x}{1-x^2}dx$$

$$=4\int_1^{3/4}\frac{-dt}{t}\ \because 1-x^2=t\Rightarrow -2xdx=dt$$

$$=4[\ln(t)]_{3/4}^{1}$$

$$=4\left[\ln(1)-\ln\left(\frac{3}{4}\right)\right]$$

$$=4\ln\left(\frac{4}{3}\right)$$

5. Let $\arg\left(\frac{z+1}{z-1}\right)=\frac{\pi}{4}$. Then the locus of z is a circle whose radius and centre respectively are:

(A) $\sqrt{2},\ (0,1)$

(B) $\sqrt{2},\ (0,-1)$

(C) $\sqrt{2},\ (0,0)$

(D) $1,\ (1,1)$

Answer: (B)

Solution:

$\arg\left(\frac{z+1}{z-1}\right)=\frac{\pi}{4}$

Let $z=x+iy$

$\Rightarrow \arg\left(\frac{x+iy+1}{x+iy-1}\right)=\frac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\frac{y}{x+1}\right)-\tan^{-1}\left(\frac{y}{x-1}\right)=\frac{\pi}{4}$

taking tan on both sides, we get

$$\frac{\frac{y}{x+1}-\frac{y}{x-1}}{1+\frac{y}{x+1}\cdot\frac{y}{x-1}}=1$$

$\Rightarrow \ \frac{-2y}{x^2-1+y^2}=1$

$\Rightarrow \ x^2-1+y^2=-2y$

$= \ x^2+y^2+2y-1=0$

$C(0,-1),\ r=\sqrt{0+1+1}=\sqrt{2}$

6. If A and B are two square matrices of order 2×2 such that $A=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$; $B=\begin{bmatrix}1 & 0\\ i & 1\end{bmatrix}$ (where $i=\sqrt{-1}$) and $A^{\top}B^{2021}A=Q$, then the value of AQ$A^{\top}$ is

(A) $\begin{bmatrix}1 & 0\\ i & 1\end{bmatrix}$

(B) $\begin{bmatrix}1 & 0\\ 2021i & 1\end{bmatrix}$

(C) $\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$

(D) $\begin{bmatrix}1 & 0\\ 2020i & 1\end{bmatrix}$

Answer: (B)

Solution:

Given,

$A=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$ $B=\begin{bmatrix}1 & 0\\ i & 1\end{bmatrix}$

$A^{\top}B^{2021}A=Q$

$B=\begin{bmatrix}1 & 0\\ i & 1\end{bmatrix}$

$B^2=\begin{bmatrix}1 & 0\\ i & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ i & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 2i & 1\end{bmatrix}$

$B^3=\begin{bmatrix}1 & 0\\ 2i & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ i & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 3i & 1\end{bmatrix}$

$B^4=\begin{bmatrix}1 & 0\\ 3i & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ i & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 4i & 1\end{bmatrix}$

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$B^{2021}=\begin{bmatrix}1 & 0\\ 2021i & 1\end{bmatrix}$

Now, given, that

$A^{\top}B^{2021}A=Q$

$AQA^{\top}=AA^{\top}B^{2021}AA^{\top}$ ...(1)

We know $A=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} \Rightarrow A^{\top}=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$

$\Rightarrow AA^{\top}=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$

So, Equation (1) will be

$AQA^{\top}=IB^{2021}I=B^{2021}$

$=\begin{bmatrix}1 & 0\\ 2021i & 1\end{bmatrix}$

7. The mean of 20 observations is 10 and standard deviation is 2.5. If one of the observation 25 is replaced by 35 then new mean and standard deviation are α and $\sqrt{\beta}$ respectively, then ordered pair (α, β) is

(A) $(10.5, 36)$

(B) $(10.5, 26)$

(C) $(10.5, 25)$

(D) $(10.5, 23)$

Answer: (B)

Solution:

Given

Mean $(\bar{x}_{old})=10$

$S.D.=2.5$

Number of observations $(n)=20$

$\bar{x}_{old}=\dfrac{x_1+x_2+x_3\ldots x_{19}+25}{20}=10$

$\Rightarrow x_1+x_2+x_3+\ldots x_{19}+25=200$

$\Rightarrow x_1+x_2+x_3+\ldots x_{19}=175$

As 25 is replaced by 35

$\bar{x}_{new}=\dfrac{x_1+x_2+\cdots x_{19}+135}{20}=\dfrac{175+135}{20}$

$= \dfrac{210}{20} = 10.5$

$(\text{S.D.})_{old} = \sqrt{\dfrac{x_1^2 + x_2^2 + x_3^2 + \ldots x_{19}^2 + (25)^2}{20} - (10)^2}$

$\Rightarrow 2.5 = \sqrt{\dfrac{x_1^2 + x_2^2 + x_3^2 + \ldots x_{19}^2 + 625}{20} - 100}$

$\Rightarrow x_1^2 + x_2^2 + x_3^2 + \ldots x_{19}^2 = 1500$

$(\text{S.D.})_{new} = \sqrt{\dfrac{x_1^2 + x_2^2 + x_3^2 + \cdots x_{19}^2 + (35)^2}{20} - (10.5)^2}$

$= \sqrt{\dfrac{1500 + 1225}{20} - 110.25}$

$= \sqrt{26}$

$\Rightarrow (\alpha, \beta) = (10.5, \sqrt{26})$

8. *If* $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}, \vec{b} = \hat{\imath} - \hat{k}, \vec{a} \times \vec{c} = \vec{b}, \vec{a} \cdot \vec{c} = 3$, then the value of $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$ is

(A) $\sqrt{2}$

(B) -2

(C) 2

(D) $-\sqrt{2}$

Answer: (B)

Solution:

$\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$

$\vec{b} = \hat{l} - \hat{k}$

$\vec{a} \times \vec{c} = \vec{b}, \vec{a} \cdot \vec{c} = 3$

Now, we know

$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \vec{a} \cdot (\vec{b} \times \vec{c})$

Using properties of scalar Triple product

We can write $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \left[\vec{b}\ \vec{c}\ \vec{a}\right] = \vec{b} \cdot (\vec{c} \times \vec{a})$

$\Rightarrow \left[\vec{a}\ \vec{b}\ \vec{c}\right] = -\vec{b} \cdot \left(\vec{a} \times \vec{c}\right) \left\{\because \vec{c} \times \vec{a} = -\vec{a} \times \vec{c}\right\}$

$= -\vec{b}.\vec{b}$

$= -(\hat{i} - \hat{k}) \cdot (\hat{i} - \hat{k})$

$= -2$

9. In a city, 89% people are suffered from diabetes and 98% people are suffered from heart diseases. If $x\%$ people are suffered from diabetes as well as heart diseases then the possible values of x cannot be lie in which of the following set:

(A) $\{81, 83, 85, 86\}$

(B) $\{82,87,88,91\}$

(C) $\{87,88,89,90\}$

(D) $\{88,89,92,95\}$

Answer: (A)

Solution:

Let

$D =$ Number of people suffered from diabetes.

$H =$ Number of people suffered from heart diseases.

Given

$n(D) = 89\%$

$n(H) = 98\%$

$n(D \cap H) = x\,\%$

We know

$n(D \cap H) = n(D) + n(H) - n(D \cup H)$

For possible range of $n(D \cap H)$

$\max\{0, n(D) + n(H) - n(D \cup H)\} \leqslant n(D \cap H) \leqslant \min\{n(D), n(H)\}$

$\max\{0, (85 + 98 - 100)\%\} \leqslant x\% \leqslant \min\{89\%, 98\%\}$

$87\% \leqslant x \leqslant 89\%$

10. If $\ell n(x + y) = 4xy$, then the value of $\frac{d^2y}{dx^2}$ at $x = 0$ is

(A) 30

(B) 40

(C) 35

(D) 38

Answer: (B)

Solution:

Given $\ell n(x + y) = 4xy \ldots$ (i)

Put $x = 0$ in (i) $\ell\text{n}y = 0 \Rightarrow y = 1$

from (i)

$\ell\text{n}(x + y) = 4xy$

$\Rightarrow x + y = e^{4xy}$

now differentiate $wrt\ x$

$$\Rightarrow 1 + \frac{dy}{dx} = e^{4xy}\left(4y + 4x\frac{dy}{dx}\right) \ldots\text{(ii)}$$

At $(0, 1)$

$$\Rightarrow 1 + \frac{dy}{dx} = e^{4xy}\left(4y + 4x\frac{dy}{dx}\right) \ldots\text{(iii)}$$

$$1 + \left(\frac{dy}{dx}\right)_{x=0} = 4 \Rightarrow \left(\frac{dy}{dx}\right) = 3 \ldots\text{(iv)}$$

Now again differentiate (ii)

$$\frac{d^2y}{dx^2} = e^{4xy}\left(4y + 4x\frac{dy}{dx}\right)^2 + e^{4xy}\left(\frac{4dy}{dx} + \frac{4dy}{dx} + \frac{4xd^2y}{dx^2}\right)$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{x=0}$$

$$= e^0\,(4 \times 1 + 4 \times 0 \times 3)^2 + e^0\left(4 \times 3 + 4 \times 3 + 4 \times 0 \times \frac{d^2y}{dx^2}\right)$$

$$= (4)^2 + 24 = 16 + 24 = 40$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=0} = 40$$

11. If locus of a variable point P, whose sum of squares of distances from points $(0,0), (0,1), (1,0)$ and $(1,1)$ is 18 , is a circle of diameter d, then value of d^2 is equal to

Answer: 16.00

Solution:

Let $P(h, k)$

Given,

Sum of square of distances of P from given points $= 18$

$$\Rightarrow (h - 0)^2 + (k - 0)^2 + (h - 0)^2 + (k - 1)^2 + (h - 1)^2 + (k - 0)^2 + (h - 1)^2 + (k - 1)^2 = 18$$

$$\Rightarrow h^2 + k^2 + h^2 + k^2 - 2k + 1 + h^2 - 2h + 1 + k^2 + h^2 - 2h + 1 + k^2 - 2k + 1 = 18$$

$$\Rightarrow 4h^2 + 4k^2 - 4h - 4k + 4 = 18$$

$$\Rightarrow h^2 + k^2 - h - k - \frac{14}{4} = 0$$

Replace $h \to x$ and $k \to y$

$$x^2 + y^2 - x - y - \frac{14}{4} = 0$$

Centre $= \left(\frac{1}{2}, \frac{1}{2}\right)$

Radius $(r) = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{14}{4}} = \sqrt{4} = 2$

$d = 2r = 2 \times 2 = 4$

So, $d^2 = 16$

12. If $(1+y)\tan^2 x + \tan x \frac{dy}{dx} + y = 0$ and $\lim_{x \to 0^+} xy = 1$, then the value of $y\left(\frac{\pi}{4}\right)$ is

(A) $\frac{\pi}{4}$

(B) 0

(C) $-\frac{\pi}{4}$

(D) $-\frac{\pi}{2}$

Answer: (A)

Solution:

$$(1+y)\tan^2 x + \tan x \frac{dy}{dx} + y = 0$$

Dividing by $\tan x$

$$\frac{dy}{dx} + (1+y)\tan x = -y\cot x$$

$\frac{dy}{dx} + (\tan x + \cot x)y = -\tan x$ ….. (i)

It's a first order linear differential equation

$$I.F. = e^{\int(\tan x + \cot x)dx}$$

$$= e^{\int \frac{\tan^2 x + 1}{\tan x}dx}$$

$$e^{\int \frac{\sec^2 x}{\tan x}dx} = e^{\ln \tan x} = \tan x$$

So equation (i) will be come

$$y\tan x = \int -\tan^2 x\, dx + C$$

$$y\tan x = \int (1 - \sec^2 x)dx + C$$

$$y\tan x = \int dx - \int \sec^2 x dx + C$$

$$y\tan x = x - \tan x + C$$

Given $\lim_{x \to 0^+} xy = 1$

$$\Rightarrow \lim_{x \to 0^+} \left(\frac{x}{\tan x}\right)(x - \tan x + C) = 1$$

$$\Rightarrow 1(0 - 0 + C) = 1$$

$\Rightarrow C = 1$

So the function is

$y \tan x = x - \tan x + 1$

$y \tan x = x - \tan x + 1$

$y\left(\frac{\pi}{4}\right) \Rightarrow y.\tan\frac{\pi}{4} = \frac{\pi}{4} - \tan\frac{\pi}{4} + 1$

$y.1 = \frac{\pi}{4} - 1 + 1$

$y = \frac{\pi}{4}$

13. How many 3 digit numbers are possible using the digits 0,1,3,4,6,7 if repetition is allowed.

Answer: 180

Solution: The hundreds place can be filled in 5 ways, tens place can be filled in 6 ways and units place can be filled in 6 ways.

$5 \times 6 \times 6 = 180$

14. If $\frac{\cos x}{1+\sin x} = |\tan x|$, then number of solutions in $[0, 2\pi]$ is

Answer: (1)

Solution:

$\frac{\cos x}{1+\sin x} = |\tan x| \ \{\sin x \neq -1\}$

Case-I $\tan x \geqslant 0$

$\Rightarrow \frac{\cos x}{1+\sin x} = \frac{\sin x}{\cos x}$

$\Rightarrow \cos^2 x = \sin x + \sin^2 x$

$\Rightarrow 2\sin^2 x + \sin x - 1 = 0$

$\Rightarrow \sin = \frac{1}{2}, -1$

$\Rightarrow \sin x = \frac{1}{2}$

$\sin x = -1$ not possible

$\Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$ (rejected)

$\Rightarrow$ 1 Solution i.e. $\frac{\pi}{6}$

Case-II $\tan x < 0$

$\frac{\cos x}{1+\sin x} = -\tan x$

$\Rightarrow \cos^2 x = -\sin x - \sin^2 x$

$\Rightarrow \sin x = -1$ (not possible)

15. A wire of length 26 units is cut into two parts which are bent respectively to form a square and a circle. If the sum of areas of square and the circle so formed is minimum, then the circumference of circle is

(A) $\frac{36\pi}{\pi+4}$

(B) $\frac{18\pi}{\pi+4}$

(C) $\frac{9\pi}{\pi+2}$

(D) $\frac{3\pi}{\pi+9}$

Answer: (A)

Solution:

Let the radius of circle is r and side of square is a.

Given, that

$2\pi r + 4a = 36$

$\Rightarrow r = \dfrac{18 - 2a}{\pi}$

Now sum of areas of circle and square

$A = \pi r^2 + a^2$

$A = \dfrac{(18-2a)^2}{\pi} + a^2$

$\dfrac{dA}{da} = \dfrac{2(18-2a)(-2)}{\pi} + 2a$

$\dfrac{dA}{da} = 0 \Rightarrow \; 36 - 4a = a\pi \Rightarrow a = \dfrac{36}{\pi + 4}$

$\dfrac{d^2A}{da^2} = \dfrac{8}{\pi} + 2 > 0$

So, the area is minimum

Now, $r = \frac{18-2a}{\pi} = \frac{18-\frac{72}{\pi+4}}{\pi} = \frac{18}{\pi+4}$

So, circumference $= 2\pi r$

$= 2\pi \times \frac{18}{\pi+4} = \frac{36\pi}{\pi+4}$ units

16. Tangent at a point P on the ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$ is perpendicular to $x + 2y = 0$. Then the value of $(5 - e^2) \times$ Area of $\Delta SPS'$ is, where e is eccentricity, S & S' are foci and P is in second quadrant

Answer: 6

Solution:

$\frac{x^2}{8}+\frac{y^2}{4}=1 \Rightarrow e=\sqrt{1-\frac{4}{8}}=\frac{1}{\sqrt{2}} \Rightarrow ae=2$

Tangent perpendicular to $x+2y=0$

$y=mx \pm \sqrt{8m^2+4} \qquad \{m=2\}$

$y=2x \pm 6$ { P in 2^{nd} quadrant)

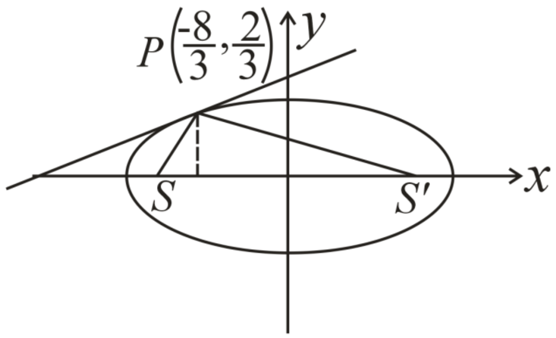
$y=2x+6 \Rightarrow 2x-y=-6$

Now comparing with

$\frac{xx_1}{8}+\frac{yy_1}{4}=1$, we get,

$\frac{x_1}{16}=\frac{y_1}{-4}=\frac{1}{-6}$

$P\left(\frac{-8}{3},\frac{2}{3}\right)$



(Area of SPS′) $\times (5-e^2)$

$=\frac{1}{2}\times 4\times \frac{2}{3}\times\left(5-\frac{1}{2}\right)$

$=6$ Sq. Units.

17. If $f(x)=\cos\left[2\tan^{-1}\left(\sin\left(\cot^{-1}\sqrt{\frac{1-x}{x}}\right)\right)\right]$, then

(A) $f'(x)(1-x)^2-2f^2(x)=0$

(B) $f'(x)(x-1)^2+2f(x)=0$

(C) $f'(x)(1-x)^2+2f^2(x)=0$

(D) $f'(x)(x+1)^2-2f(x)=0$

Answer: (C)

Solution: Let $\cot^{-1}\sqrt{\frac{1-x}{x}}=\theta \Rightarrow \cot\theta=\sqrt{\frac{1-x}{x}} \Rightarrow \sin\theta=\sqrt{x}$

Now, $f(x)=\cos\,[2\tan^{-1}(\sin\theta)]$

$f(x)=\cos\left[2\tan^{-1}\sqrt{x}\right]$

Let $\tan^{-1}\sqrt{x}=\alpha$

$\Rightarrow \sqrt{x} = \tan\alpha$

So, $f(x) = \cos(2\alpha)$

$f(x) = \frac{1-x}{1+x}$

$f'(x) = \frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2}$

$f'(x) = \frac{-2}{(1+x)^2}$

Multiplying by $(1-x)^2$, we get

$$f'(x)(1-x)^2 = \frac{-2}{(1+x)^2}(1-x)^2$$

$$f'(x)(1-x)^2 = -2f^2(x)$$

18. Find the value of $\frac{1}{1+x} + \frac{2}{1+x^2} + \frac{2^2}{1+x^4} + \cdots + \frac{2^{100}}{1+x^{200}}$ at $x = 2$

(A) $\frac{2^{200}}{1+2^{400}} - 1$

(B) $\frac{2^{101}}{1-2^{400}} + 1$

(C) $\frac{2^{100}}{1-2^{400}} - 1$

(D) $\frac{2^{100}}{1+2^{200}} - 1$

Answer (B)

Solution: Given $\frac{1}{1+x} + \frac{2}{1+x^2} + \frac{2^2}{1+x^4} + \cdots + \frac{2^{100}}{1+x^{200}}$

add and subtract $\frac{1}{1-x}$, we get

$\frac{1}{1-x} + \frac{1}{1+x} + \frac{2}{1+x^2} + \frac{2^2}{1+x^4} + \cdots \frac{2^{100}}{1+x^{200}} - \frac{1}{1-x}$

$= \frac{2}{1-x^2} + \frac{2}{1+x^2} + \frac{2^2}{1+x^4} \ldots\ldots \frac{2^{10}}{1+x^{200}} - \frac{1}{1-x}$

$= \frac{2^2}{1-x^4} + \frac{2^2}{1+x^4} + \ldots\ldots \frac{2^{100}}{1+x^{200}} - \frac{1}{1-x}$

similarly we will get

$\frac{2^{101}}{1-x^{400}} - \frac{1}{1-x}$

At $x = 2$, we get,

$\frac{2^{101}}{1-2^{400}} + 1$

19. A circle with centre $(-15,0)$ and radius is equal to $\frac{15}{2}$. Chord to circle through $(-30,0)$ is tangent to $y^2 = 30x$. Find the length of the chord.

(A) $\frac{15}{\sqrt{5}}$

(B) $\frac{15}{4}$

(C) $\frac{15}{\sqrt{3}}$

(D) $\frac{15}{\sqrt{7}}$

Answer (A)

Solution: Given

$y^2 = 30x$

$4a = 30 \Rightarrow a = \frac{15}{2}$

Equation of tangent

$y = mx + \frac{a}{m}$

$y = mx + \frac{15}{2m}$

Passes through $(-30,0)$

$\Rightarrow 0 = -30m + \frac{15}{2m}$

$\Rightarrow 30m = \frac{15}{2m} \Rightarrow m^2 = \frac{1}{4} \Rightarrow m = \pm\frac{1}{2}$

Case I- $m = \frac{1}{2}$

$y = \frac{x}{2} + 15$

Distance of line from centre is

$$d_1 = \frac{\left|-\frac{15}{2}+15\right|}{\sqrt{1^2+\left(\frac{1}{2}\right)^2}} = \frac{\frac{15}{2}}{\frac{\sqrt{5}}{2}} = \frac{15}{\sqrt{5}}$$

Case$-II$ $m = -\frac{1}{2}$

$y = -\frac{x}{2} - 15$

Distance of line from centre is

$$d_2 = \frac{\left|\frac{15}{2}-15\right|}{\sqrt{1^2+\left(\frac{1}{2}\right)^2}} = \frac{15}{\sqrt{5}}$$

$\Rightarrow d_1 = d_2 = d = \frac{15}{\sqrt{5}}$

Length of chord $= 2\sqrt{r^2 - d^2}$

$= 2\sqrt{\left(\frac{15}{2}\right)^2 - \left(\frac{15}{\sqrt{5}}\right)^2}$

$= 2 \times 15\sqrt{\frac{1}{4} - \frac{1}{5}}$

$= 30 \times \sqrt{\frac{1}{20}}$

$= \frac{30}{\sqrt{20}} = \frac{15}{\sqrt{5}}$

20. The value of $\lim_{n\to\infty} \frac{1}{n}\sum_{r=0}^{2n-1} \frac{n^2}{n^2+4r^2}$ is

(A) $\tan^{-1}4$

(B) $\tan^{-1}2$

(C) $\frac{1}{2}\tan^{-1}4$

(D) $\frac{1}{2}\tan^{-1}2$

Answer: (C)

Solution:

$\frac{1}{n}\sum_{r=0}^{2n-1} \frac{1}{1+4\left(\frac{r}{n}\right)^2} \cdot \frac{1}{n}$

$= \int_0^2 \frac{1}{1+4x^2} dx$ let $\frac{r}{n} = x$, $\frac{1}{n} = dx$

$= \frac{1}{2}[\tan^{-1}(2x)]_0^2$

$= \frac{1}{2}[\tan^{-1}4 - \tan^{-1}0]$

$= \frac{1}{2}\tan^{-1} 4$

21. If $z = \frac{-1+\sqrt{3}i}{2}$, then the value of $21 + \left(z + \frac{1}{z}\right)^3 + \left(z^2 + \frac{1}{z^2}\right)^3 + \left(z^3 + \frac{1}{z^3}\right)^3 \dots. + \left(z^{21} + \frac{1}{z^{21}}\right)^3$ is equal to

Ans: (63)

Solution: Here, $z = \omega$, so $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$.

Now, $z + \frac{1}{z} = \omega + \frac{1}{\omega} = \omega + \omega^2 = -1$

Similarly, $z^2 + \frac{1}{z^2} = \omega^2 + \frac{1}{\omega^2} = \omega + \omega^2 = -1$, $z^3 + \frac{1}{z^3} = \omega^3 + \frac{1}{\omega^3} = 2$

Hence, $21 + \left(z + \frac{1}{z}\right)^3 + \left(z^2 + \frac{1}{z^2}\right)^3 + \left(z^3 + \frac{1}{z^3}\right)^3 \dots. + \left(z^{21} + \frac{1}{z^{2}1}\right)^3 = 21 + (-1 - 1 + 8) + 6\ (6\text{ times}) = 63$