CBSE NCERT Solutions for Class 12 Maths Chapter 05

Back of Chapter Questions

Exercise 5.1

1. Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$, at $x = -3$ and at $x = 5$.

Solution:

Given function is $f(x) = 5x - 3$

At $x = 0, f(0) = 5(0) - 3 = -3$

$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (5x - 3) = -3$

$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (5x - 3) = -3$

Here, at $x = 0, \text{LHL} = \text{RHL} = f(0) = -3$

Hence, the function f is continuous at $x = 0$.

Now at $x = -3, f(-3) = 5(-3) - 3 = -18$

$\text{LHL} = \lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (5x - 3) = -18$

$\text{RHL} = \lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} (5x - 3) = -18$

Here, at $x = -3, \text{LHL} = \text{RHL} = f(-3) = -18$

Hence, the function f is continuous at $x = -3$.

At $x = 5, f(5) = 5(5) - 3 = 22$

$LHL = \lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (5x - 3) = 22$

$\text{RHL} = \lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} (5x - 3) = 22$

Here, at $x = 5, \text{LHL} = \text{RHL} = f(5) = 22$

Hence, the function f is continuous at $x = 5$.

2. Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$.

Solution:

Given function is $f(x) = 2x^2 - 1$.

At $x = 3, f(3) = 2(3)^2 - 1 = 17$

$\text{LHL} = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(2x^2 - 1) = 17$

$\text{RHL} = \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(2x^2 - 1) = 17$

Here, at $x = 3, \text{LHL} = \text{RHL} = f(3) = 17$

Therefore, the function f is continuous at $x = 3$.

3. Examine the following functions for continuity:

(a) $f(x) = x - 5$

(b) $f(x) = \frac{1}{x-5}, x \neq 5$

(c) $f(x) = \frac{x^2-25}{x+5}, x \neq -5$

(d) $f(x) = |x - 5|$

Solution:

(a) Given function $f(x) = x - 5$

Let k be any real number. At $x = k, f(k) = k - 5$

$\text{LHL} = \lim_{x\to k^-} f(x) = \lim_{x\to k^-}(x - 5) = k - 5$

$\text{RHL} = \lim_{x\to k^+} f(x) = \lim_{x\to k^+}(x - 5) = k - 5$

At $x = k, \text{LHL} = \text{RHL} = f(k) = k - 5$

Hence, the function f is continuous for all real numbers.

(b) Given function $f(x) = \frac{1}{x-5}, x \neq 5$

Let $k(k \neq 5)$ be any real number. At $x = k, f(k) = \frac{1}{k-5}$

$\text{LHL} = \lim_{x\to k^-} f(x) = \lim_{x\to k^-}\left(\frac{1}{x-5}\right) = \frac{1}{k-5}$

$\text{RHL} = \lim_{x\to k^+} f(x) = \lim_{x\to k^+}\left(\frac{1}{x-5}\right) = \frac{1}{k-5}$

At $x = k, \text{LHL} = \text{RHL} = f(k) = \frac{1}{k-5}$

Hence, the function f is continuous for all real numbers (except 5).

(c) Given function $f(x) = \frac{x^2-25}{x+5}, x \neq -5$

Let $k(k \neq -5)$ be any real number.

At $x = k, f(k) = \frac{k^2-25}{k+5} = \frac{(k+5)(k-5)}{(k+5)} = (k+5)$

$\text{LHL} = \lim_{x\to k^-} f(x) = \lim_{x\to k^-}\left(\frac{x^2-25}{x+5}\right) = \lim_{x\to k^-}\left(\frac{(k+5)(k-5)}{(k+5)}\right) = k+5$

$\text{RHL} = \lim_{x\to k^+} f(x) = \lim_{x\to k^+}\left(\frac{x^2-25}{x+5}\right) = \lim_{x\to k^+}\left(\frac{(k+5)(k-5)}{(k+5)}\right) = k+5$

At $x = k, \text{LHL} = \text{RHL} = f(k) = k+5$

Hence, the function f is continuous for all real numbers (except -5).

(d) Given function is $f(x) = |x-5| = \begin{cases} 5-x, & x < 5 \\ x-5, & x \geq 5 \end{cases}$

Let k be any real number. According to question, k can be $k < 5$ or $k = 5$ or $k > 5$.

First case: If $k < 5$,

$f(k) = 5-k$ and $\lim_{x\to k} f(x) = \lim_{x\to k}(5-x) = 5-k$. Here, $\lim_{x\to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers less than 5.

Second case: If $k = 5$,

$f(k) = k-5$ and $\lim_{x\to k} f(x) = \lim_{x\to k}(x-5) = k-5$. Here, $\lim_{x\to k} f(x) = f(k)$

Hence, the function f is continuous at $x = 5$.

Third case: If $k > 5$,

$f(k) = k-5$ and $\lim_{x\to k} f(x) = \lim_{x\to k}(x-5) = k-5$. Here, $\lim_{x\to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers greater than 5.

Hence, the function f is continuous for all real numbers.

4. Prove that the function $f(x) = x^n$, is continuous at $x = n$, where n is a positive integer.

Solution:

Given function is $f(x) = x^n$.

At $x = n, f(n) = x^n$

$\lim_{x\to n} f(x) = \lim_{x\to n}(x^n) = x^n$

Here, at $x = n, \lim_{x\to n} f(x) = f(n) = x^n$

Since $\lim_{x\to n} f(x) = f(n) = x^n$

Hence, the function f is continuous at $x = n$, where n is positive integer.

5. Is the function f defined by $f(x) = \begin{cases} x, & x \leq 1 \\ 5, & x > 1 \end{cases}$

continuous at $x = 0$? At $x = 1$? At $x = 2$?

Solution:

Given function is $f(x) = \begin{cases} x, & x \leq 1 \\ 5, & x > 1 \end{cases}$

At $x = 0, f(0) = 0$

$\lim_{x \to 0} f(x) = \lim_{x \to 0}(x) = 0$

Here at $x = 0, \lim_{x \to 0} f(x) = f(0) = 0$

Hence, the function f is discontinuous at $x = 0$.

At $x = 1, f(1) = 1$

$\text{LHL} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(x) = 1$

$\text{RHL} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+}(5) = 5$

Here, at $x = 1, \text{LHL} \neq \text{RHL}$.

Hence, the function f is discontinuous at $x = 1$.

At $x = 2, f(2) = 5$

$\lim_{x \to 2} f(x) = \lim_{x \to 2}(5) = 5$

Here, at $x = 2, \lim_{x \to 2} f(x) = f(2) = 5$

Hence, the function f is continuous at $x = 2$.

6. Find all points of discontinuity of f, where f is defined by $f(x) = \begin{cases} 2x + 3, & \text{If } x \leq 2 \\ 2x - 3, & \text{If } x > 2 \end{cases}$

Solution:

Let k be any real number. According to question, $k < 2$ or $k = 2$ or $k > 2$

First case: $k < 2$

$f(k) = 2k + 3$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(2x + 3) = 2k + 3$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers smaller than 2.

Second case: If $k = 2, f(2) = 2k + 3$

$\text{LHL} = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x + 3) = 7$

$\text{RHL} = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x - 3) = 1$

Here, at $x = 2$, LHL $\neq$ RHL. Hence, the function f is discontinuous at $x = 2$.

Third case: If $k > 2$,

$f(k) = 2k - 3$ and $\lim_{x \to k} f(x) = \lim_{x \to k} (2x - 3) = 2k - 3$. Here, $\lim_{x \to k} f(x) = f(k)$

Therefore, the function f is continuous for all real numbers greater than 2.

Hence, the function f is discontinuous only at $x = 2$.

7. Find all points of discontinuity of f,

where f is defined by $f(x) = \begin{cases} |x| + 3, & \text{If } x \leq -3 \\ -2x, & \text{If } -3 < x < 3 \\ 6x + 2, & \text{If } x \geq 3 \end{cases}$

Solution:

Let k be any real number. According to question,

$k < -3$ or $k = -3$ or $-3 < k < 3$ or $k = 3$ or $k > 3$

First case: If $k < -3$,

$f(x) = -k + 3$ and $\lim_{x \to k} f(x) = \lim_{x \to k} (-x + 3) = -k + 3$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers less than -3.

Second case: If $k = -3, f(-3) = -(-3) + 3 = 6$

$\text{LHL} = \lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (-x + 3) = -(-3) + 3 = 6$

$\text{RHL} = \lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} (-2x) = -2(-3) = 6$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous at $x = -3$.

Third case: If $-3 < k < 3$,

$f(k) = -2k$ and $\lim_{x \to k} f(x) = \lim_{x \to k} f(-2x) = -2k$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous at $-3 < x < 3$.

Fourth case: If $k = 3$,

$\text{LHL} = \lim_{x \to k^-} f(x) = \lim_{x \to k^-} (-2x) = -2k$

$\text{RHL} = \lim_{x \to k^+} f(x) = \lim_{x \to k^+} (6x + 2) = 6k + 2,$

Here, at $x = 3$, LHL $\neq$ RHL. Hence, the function f is discontinuous at $x = 3$.

Fifth case: If $k > 3$,

$f(k) = 6k + 2$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(6x + 2) = 6k + 2$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all numbers greater than 3.

Hence, the function f is discontinuous only at $x = 3$.

8. Find all points of discontinuity of f, where f is defined by $f(x) = \begin{cases} \frac{|x|}{x}, & \text{If } x \neq 0 \\ 0, & \text{If } x = 0 \end{cases}$

Solution:

After redefining the function f, we get

$$f(x) = \begin{cases} -\frac{x}{x} = -1, & \text{If } x < 0 \\ 0, & \text{If } x = 0 \\ \frac{x}{x} = 1, & \text{If } x > 0 \end{cases}$$

Let k be any real number. According to question, $k < 0$ or $k = 0$ or $k > 0$.

First case: If $k < 0$,

$f(k) = -\frac{k}{k} = -1$ and $\lim_{x \to k} f(x) = \lim_{x \to k}\left(-\frac{x}{x}\right) = -1$. Hence, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers smaller than 0.

Second case: If, $k = 0, f(0) = 0$

$\text{LHL} = \lim_{x \to k^-} f(x) = \lim_{x \to k^-}\left(-\frac{x}{x}\right) = -1$ and $\text{RHL} = \lim_{x \to k^+} f(x) = \lim_{x \to k^+}\left(\frac{x}{x}\right) = 1$.

Here, at $x = 0, \text{LHL} \neq \text{RHL}$. Hence, the function f is discontinuous at $x = 0$.

Third case: If $k > 0$,

$f(k) = \frac{k}{k} = 1$ and $\lim_{x \to k} f(x) = \lim_{x \to k}\left(\frac{x}{x}\right) = 1$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers greater than 0.

Therefore, the function f is discontinuous only at $x = 0$.

9. Find all points of discontinuity of f, where f is defined by $f(x) = \begin{cases} \frac{x}{|x|}, & \text{If } x < 0 \\ -1, & \text{If } x \geq 0 \end{cases}$

Solution:

Redefining the function, we get

$$f(x) = \begin{cases} \frac{x}{|x|} = \frac{x}{-x} = -1, & \text{If } x < 0 \\ -1, & \text{If } x \geq 0 \end{cases}$$

Here, $\lim_{x \to k} f(x) = f(k) = -1$, where k is a real number.

Hence, the function f is continuous for all real numbers.

10. Find all points of discontinuity of f, where f is defined by $f(x) = \begin{cases} x + 1, & \text{If} \geq 1 \\ x^2 + 1, & \text{If } x < 1 \end{cases}$

Solution:

Given function is defined by $f(x) = \begin{cases} x + 1, & \text{If} \geq 1 \\ x^2 + 1, & \text{If } x < 1 \end{cases}$

Let k be any real number. According to question, $k < 1$ or $k = 1$ or $k > 1$

First case: If $k < 1$,

$f(k) = k^2 + 1$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(x^2 + 1) = k^2 + 1$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers smaller than 1.

Second case: If $k = 1, f(1) = 1 + 1 = 2$

$\text{LHL} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 1) = 1 + 1 = 2$

$\text{RHL} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x + 1) = 1 + 1 = 2,$

Here, at $x = 1, \text{LHL} = \text{RHL} = f(1)$. Hence, the function f is continuous at $x = 1$.

Third case: If $k > 1$,

$f(k) = k + 1$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(x + 1) = k + 1$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers greater than 1.

Therefore, the function f is continuous for all real numbers.

11. Find all points of discontinuity of f, where f is defined by $f(x) = \begin{cases} x^3 - 3, & \text{If } x \leq 2 \\ x^2 + 1, & \text{If } x > 2 \end{cases}$

Solution:

Given function is defined by $f(x) = \begin{cases} x^3 - 3, & \text{If } x \le 2 \\ x^2 + 1, & \text{If } x > 2 \end{cases}$

Let k be any real number. According to question, $k < 2$ or $k = 2$ or $k > 2$

First case: If $k < 2$,

$f(k) = k^3 - 3$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(x^3 - 3) = k^3 - 3$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers less than 2.

Second case: If $k = 2, f(2) = 2^3 - 3 = 5$

$\text{LHL} = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^3 - 3) = 2^3 - 3 = 5$

$\text{RHL} = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 + 1) = 2^2 + 1 = 5$

Here at $x = 2, \text{LHL} = \text{RHL} = f(2)$

Hence, the function f is continuous at $x = 2$.

Third case: If $k > 2$,

$f(k) = k^2 + 1$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(x^2 + 1) = k^2 + 1$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for real numbers greater than 2.

Hence, the function f is continuous for all real numbers.

12. Find all points of discontinuity of f, where f is defined by $f(x) = \begin{cases} x^{10} - 1, & \text{If } x \le 1 \\ x^2, & \text{If } x > 1 \end{cases}$

Solution:

Given function is defined by $f(x) = \begin{cases} x^{10} - 1, & \text{If } x \le 1 \\ x^2, & \text{If } x > 1 \end{cases}$

Let k be any real number. According to question, $k < 1$ or $k = 1$ or $k > 1$

First case: If $k < 1$,

$f(k) = k^{10} - 1$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(x^{10} - 1) = k^{10} - 1$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers less than 1.

Second case: If $k = 1, f(1) = 1^{10} - 1 = 0$

$\text{LHL} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^{10} - 1) = 0$

$\text{RHL} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2) = 1$

Here at $x = 1, \text{LHL} \ne \text{RHL}$. Hence, the function f is discontinuous at $x = 1$.

Third case: If $k > 1$,

$f(k) = k^2$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(x^2) = k^2$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real values greater than 1.

Hence, the function f is discontinuous only at $x = 1$.

13. Is the function defined by $f(x) = \begin{cases} x+5, & \text{If } x \le 1 \\ x-5, & \text{If } x > 1 \end{cases}$ a continuous function?

Solution:

Given function is defined by $f(x) = \begin{cases} x+5, & \text{If } x \le 1 \\ x-5, & \text{If } x > 1 \end{cases}$

Let, k be any real number. According to question, $k < 1$ or $k = 1$ or $k > 1$

First case: If $k < 1$,

$f(k) = k + 5$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(x + 5) = k + 5$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers less than 1.

Second case: If $k = 1, f(1) = 1 + 5 = 6$

$\text{LHL} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(x + 5) = 6$

$\text{RHL} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+}(x - 5) = -4,$

Here at $x = 1, \text{LHL} \ne \text{RHL}$. Hence, the function f is discontinuous at $x = 1$.

Third case: If $k > 1$,

$f(k) = k - 5$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(x - 5) = k - 5$

Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers greater than 1.

Therefore, the function f is discontinuous only at $x = 1$.

14. Discuss the continuity of the function f,

where f is defined by $f(x) = \begin{cases} 3, & \text{If } 0 \le x \le 1 \\ 4, & \text{If } 1 < x < 3 \\ 5, & \text{If } 3 \le x \le 10 \end{cases}$

Solution:

Given function is defined by $f(x) = \begin{cases} 3, & \text{If } 0 \le x \le 1 \\ 4, & \text{If } 1 < x < 3 \\ 5, & \text{If } 3 \le x \le 10 \end{cases}$

Let k be any real number. According to question, k can be

$0 \le k \le 1$ or $k = 1$ or $1 < k < 3$ or $k = 3$ or $3 \le k \le 10$

First case: If $0 \le k \le 1$,

$f(k) = 3$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(3) = 3$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for $0 \le x \le 1$.

Second case: If $k = 1, f(1) = 3$

$\text{LHL} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(3) = 3$

$\text{RHL} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+}(4) = 4,$

Here at $x = 1$, LHL $\neq$ RHL. Hence, the function f is discontinuous at $x = 1$.

Third case: If $1 < k < 3$,

$f(k) = 4$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(4) = 4$. Here $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for $1 < x < 3$.

Fourth case: If $k = 3$,

$\text{LHL} = \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-}(4) = 4$ and $\text{RHL} = \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+}(5) = 5,$

Here at $x = 3$, LHL $\neq$ RHL. Hence, the function f is discontinuous at $x = 3$.

Fifth case: If $3 \le k \le 10$,

$f(k) = 5$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(5) = 5$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for $3 \le x \le 10$.

Hence, the function f is discontinuous only at $x = 1$ and $x = 3$.

15. Discuss the continuity of the function f,

where f is defined by $f(x) = \begin{cases} 2x, & \text{If } x < 0 \\ 0, & \text{If } 0 \le x \le 1 \\ 4x, & \text{If } x > 1 \end{cases}$

Solution:

Given function is defined by $f(x) = \begin{cases} 2x, & \text{If } x < 0 \\ 0, & \text{If } 0 \le x \le 1 \\ 4x, & \text{If } x > 1 \end{cases}$

Let k be any real number. According to question,

$k < 0$ or $k = 0$ or $0 \leq k \leq 1$ or $k = 1$ or $k > 1$

First case: If $k < 0$,

$f(k) = 2k$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(2x) = 2k$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers less than 0.

Second case: If $k = 0, f(0) = 0$

$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}(2x) = 0$

$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+}(0) = 0$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous at $x = 0$.

Third case: If $0 \leq k \leq 1$,

$f(k) = 0$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(0) = 0$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous at $0 \leq x \leq 1$.

Fourth case: If $k = 1$,

$\text{LHL} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(0) = 0$

$\text{RHL} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+}(4x) = 4$,

Here, at $x = 1, \text{LHL} \neq \text{RHL}$.

Hence, the function f is discontinuous at $x = 1$.

Fifth case: If $k > 1$,

$f(k) = 4k$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(4x) = 4k$.

Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers greater than 1.

Hence, the function f is discontinuous only at $x = 1$.

16. Discuss the continuity of the function f,

where f is defined by $f(x) = \begin{cases} -2, & \text{If } x \leq -1 \\ 2x, & \text{If } -1 < x \leq 1 \\ 2, & \text{If } x > 1 \end{cases}$

Solution:

Given function is defined by $f(x) = \begin{cases} -2, & \text{If } x \leq -1 \\ 2x, & \text{If } -1 < x \leq 1 \\ 2, & \text{If } x > 1 \end{cases}$

Let k be any real number

According to question, $k < -1$ or $k = -1$ or $-1 < x \leq 1$ or $k = 1$ or $k > 1$

First case: If $k < -1$,

$f(k) = -2$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(-2) = -2$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers less than -1.

Second case: If $k = -1, f(-1) = -2$

$\text{LHS} = \lim_{x \to -1^-} f(x) = \lim_{x \to -1^-}(-2) = -2$

$\text{RHL} = \lim_{x \to -1^+} f(x) = \lim_{x \to -1^+}(2x) = -2$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous at $x = -1$.

Third case: If $-1 < x \leq 1$,

$f(k) = 2k$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(2x) = 2k$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous at $-1 < x \leq 1$.

Fourth case: If $k = 1$,

$\text{LHL} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(2x) = 2$

$\text{RHL} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+}(2) = 2$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous at $x = 1$.

Fifth case: If $k > 1$,

$f(k) = 2$ and $\lim_{x \to k} f(x) = \lim_{x \to k}(2) = 2$.

Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers greater than 1.

Therefore, the function f is continuous for all real numbers.

17. Find the relationship between a and b so that the function f defined by

$f(x) = \begin{cases} ax + 1, & \text{If } x \leq 3 \\ bx + 3, & \text{If } x > 3 \end{cases}$ is continuous at $x = 3$.

Solution:

Given functions is defined by $f(x) = \begin{cases} ax + 1, & \text{If } x \leq 3 \\ bx + 3, & \text{If } x > 3 \end{cases}$

Given that the function is continuous at $x = 3$. Therefore, $\text{LHL} = \text{RHL} = f(3)$

$\Rightarrow \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$

$\Rightarrow \lim_{x \to 3^-} ax + 1 = \lim_{x \to 3^+} bx + 3 = 3a + 1$

$\Rightarrow 3a + 1 = 3b + 3 = 3a + 1$

$\Rightarrow 3a = 3b + 2 \quad \Rightarrow a = b + \frac{2}{3}$

Hence, the relationship between a and b is $a = b + \frac{2}{3}$

18. For what value of λ is the function defined by

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{If } x \leq 0 \\ 4x + 1, & \text{If } x > 0 \end{cases}$$

Continuous at $x = 0$? What about continuity at $x = 1$?

Solution:

Given function is defined as $f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{If } x \leq 0 \\ 4x + 1, & \text{If } x > 0 \end{cases}$

Given that the function is continuous at $x = 0$. Therefore, $\text{LHL} = \text{RHL} = f(0)$

$\Rightarrow \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

$\Rightarrow \lim_{x \to 0^-} \lambda(x^2 - 2x) = \lim_{x \to 0^+} 4x + 1 = \lambda[(0)^2 - 2(0)]$

$\Rightarrow \lambda[(0)^2 - 2(0)] = 4(0) + 1 = \lambda(0)$

$\Rightarrow 0.\lambda = 1 \quad \Rightarrow \lambda = \frac{1}{0}$

Hence, there is no real value of λ for which the given function be continuous.

If $x = 1$,

$f(1) = 4(1) + 1 = 5$ and $\lim_{x \to 1} f(x) = \lim_{x \to 1} 4(1) + 1 = 5$. Here, $\lim_{x \to 1} f(x) = f(1)$

Therefore, the function f is continuous for all real values of λ.

19. Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points.

Here $[x]$ denotes the greatest integer less than or equal to x.

Solution:

Given function is defined by $g(x) = x - [x]$

Let k be any integer

$\text{LHL} = \lim_{x \to k^-} f(x) = \lim_{x \to k^-} x - [x] = k - (k - 1) = 1$

$\text{RHL} = \lim_{x\to k^+} f(x) = \lim_{x\to k^+} x - [x] = k - (k) = 0,$

Here, at $x = k, \text{LHL} \neq \text{RHL}.$

Therefore, the function f is discontinuous for all integers.

20. Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$.

Solution:

Given function is defined by $f(x) = x^2 - \sin x + 5,$

At $x = \pi, f(\pi) = \pi^2 - \sin \pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$

$\lim_{x\to n} f(x) = \lim_{x\to n} x^2 - \sin x + 5 = \pi^2 - \sin \pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$

Here, at $x = \pi, \lim_{x\to n} f(x) = f(\pi) = \pi^2 + 5$

Therefore, the function $f(x)$ is continuous at $x = \pi$.

21. Discuss the continuity of the following functions:

(a) $f(x) = \sin x + \cos x$

(b) $f(x) = \sin x - \cos x$

(c) $f(x) = \sin x . \cos x$

Solution:

Let $g(x) = \sin x$

Let k be any real number. At $x = k, g(k) = \sin k$

$\text{LHL} = \lim_{x\to k^-} g(x) = \lim_{x\to k^-} \sin x = \lim_{h\to 0} \sin(k - h) = \lim_{h\to 0} \sin k \cos h - \cos k \sin h = \sin k$

$\text{RHL} = \lim_{x\to k^+} g(x) = \lim_{x\to k^+} \sin x = \lim_{h\to 0} \sin(k + h) = \lim_{h\to 0} \sin k \cos h + \cos k \sin h = \sin k$

Here, at $x = k, \text{LHL} = \text{RHL} = g(k).$

Hence, the function g is continuous for all real numbers.

Let $h(x) = \cos x$

Let k be any real number. $x = k, h(k) = \cos k$

$LHL = \lim_{x\to k^-} h(x) = \lim_{x\to k^-} \cos x = \lim_{h\to 0} \cos(k - h) = \lim_{h\to 0} \cos k \cos h + \sin k \sin h = \cos k$

$\text{RHL} = \lim_{x\to k^+} h(x) = \lim_{x\to k^+} \cos x = \lim_{h\to 0} \cos(k + h) = \lim_{h\to 0} \cos k \cos h - \sin k \sin h = \cos k$

Here, at $x=k$, $\text{LHL}=\text{RHL}=h(k)$.

Hence, the function h is continuous for all real numbers.

We know that if g and h are two continuous functions, then the functions $g+h, g-h$ and gh also be a continuous function.

Therefore, (a) $f(x)=\sin x+\cos x$ (b) $f(x)=\sin x-\cos x$ and

(c) $f(x)=\sin x.\cos x$ are continuous functions.

22. Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

Solution:

Let $g(x)=\sin x$

Let k be any real number. At $x=k, g(k)=\sin k$

$\text{LHL}=\lim_{x\to k^-} g(x)=\lim_{x\to k^-}\sin x=\lim_{h\to 0}\sin(k-h)=\lim_{h\to 0}\sin k\cos h-\cos k\sin h=\sin k$

$\text{RHL}=\lim_{x\to k^+} g(x)=\lim_{x\to k^+}\sin x=\lim_{h\to 0}\sin(k+h)=\lim_{h\to 0}\sin k\cos h+\cos k\sin h=\sin k$

Here, at $x=k$, $\text{LHL}=\text{RHL}=g(k)$.

Hence, the function g is continuous for all real numbers.

Let $h(x)=\cos x$

Let k be any real number. At $x=k, h(k)=\cos k$

$LHL=\lim_{x\to k^-} h(x)=\lim_{x\to k^-}\cos x=\lim_{h\to 0}\cos(k-h)=\lim_{h\to 0}\cos k\cos h+\sin k\sin h=\cos k$

$\text{RHL}=\lim_{x\to k^+} h(x)=\lim_{x\to k^+}\cos x=\lim_{h\to 0}\cos(k+h)=\lim_{h\to 0}\cos k\cos h-\sin k\sin h=\cos k$

Here, at $x=k$, $\text{LHL}=\text{RHL}=h(k)$.

Hence, the function h is continuous for all real numbers.

We know that if g and h are two continuous functions, then the functions $\frac{g}{h}, h\neq 0, \frac{1}{h}, h\neq 0$ and $\frac{1}{g}, g\neq 0$ are continuous functions.

Therefore, $\operatorname{cosec} x=\frac{1}{\sin x}, \sin x\neq 0$ is continuous $\Rightarrow x\neq n\pi (n\in Z)$ is continuous.

Hence, $\operatorname{cosec} x$ is continuous except $x=n\pi (n\in Z)$.

$\sec x=\frac{1}{\cos x}, \cos x\neq 0$ is continuous. $\Rightarrow x\neq\frac{(2n+1)\pi}{2}(n\in Z)$ is continuous.

Hence, $\sec x$ is continuous except $x=\frac{(2n+1)\pi}{2}(n\in Z)$.

$\cot x=\frac{\cos x}{\sin x}, \sin x\neq 0$ is continuous. $\Rightarrow x\neq n\pi (n\in Z)$ is continuous.

Hence, $\cot x$ is continuous except $x = n\pi (n \in Z)$.

23. Find all points of discontinuity of f, where $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{If } x < 0 \\ x + 1, & \text{If } x \geq 0 \end{cases}$

Solution:

Given function is defined by $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{If } x < 0 \\ x + 1, & \text{If } x \geq 0 \end{cases}$

Let k be any real number. According to question, $k < 0$ or $k = 0$ or $k > 0$

First case: If $k < 0$

$f(k) = \frac{\sin k}{k}$ and $\lim_{x \to k} f(x) = \lim_{x \to k} \left(\frac{\sin x}{x}\right) = \frac{\sin k}{k}$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers less than 0.

Second case: If $k = 0$

$f(0) = 0 + 1 = 1$

$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x + 1) = 0 + 1 = 1$

$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x + 1) = 0 + 1 = 1,$

Here at $x = 0, \text{LHL} = \text{RHL} = f(0)$. Hence, the function f is continuous at $x = 0$.

Third case: If $k > 0$

$f(k) = k + 1$ and $\lim_{x \to k} f(x) = \lim_{x \to k} (x + 1) = k + 1$. Here, $\lim_{x \to k} f(x) = f(k)$

Hence, the function f is continuous for all real numbers greater than 0.

Therefore, the function f is continuous for all real numbers.

24. Determine if f defined by

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{If } x \neq 0 \\ 0, & \text{If } x = 0 \end{cases}$$

is a continuous function?

Solution:

Given function is defined by $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{If } x \neq 0 \\ 0, & \text{If } x = 0 \end{cases}$

Let k be any real number. According to question, $k \neq 0$ or $k = 0$

First case: If $k \neq 0$

$f(k) = k^2 \sin\frac{1}{k}$ and $\lim_{x\to k} f(x) = \lim_{x\to k}\left(x^2 \sin\frac{1}{x}\right) = k^2 \sin\frac{1}{k}$. Here, $\lim_{x\to k} f(x) = f(k)$

Hence, the function f is continuous for $k \neq 0$.

Second case: If, $k = 0, f(0) = 0$

$$\text{LHL} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right)$$

We know that, $-1 \leq \sin\frac{1}{x} \leq 1, x \neq 0 \quad \Rightarrow -x^2 \leq \sin\frac{1}{x} \leq x^2$

$$\Rightarrow \lim_{x\to 0}(-x^2) \leq \lim_{x\to 0} \sin\frac{1}{x} \leq \lim_{x\to 0} x^2$$

$$\Rightarrow 0 \leq \lim_{x\to 0} \sin\frac{1}{x} \leq 0 \quad \Rightarrow \lim_{x\to 0} \sin\frac{1}{x} = 0 \quad \Rightarrow \lim_{x\to 0^-} x^2 \sin\frac{1}{x} = 0 \quad \Rightarrow \lim_{x\to 0^-} f(x) = 0$$

Similarly, $\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$

Here, at $x = 0, \text{LHL} = \text{RHL} = f(0)$

Hence, at $x = 0, f$ is continuous.

Therefore, the function f is continuous for all real numbers.

25. Examine the continuity of f, where f is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & \text{If } x \neq 0 \\ -1, & \text{If } x = 0 \end{cases}$$

Solution:

Given function is defined by $f(x) = \begin{cases} \sin x - \cos x, & \text{If } x \neq 0 \\ -1, & \text{If } x = 0 \end{cases}$

Let k be any real number. According to question, $k \neq 0$ or $k = 0$

First case: If $k \neq 0, f(0) = 0 - 1 = -1$

$$\text{LHL} = \lim_{k\to 0^-} f(x) = \lim_{k\to 0^-}(\sin x - \cos x) = 0 - 1 = -1$$

$$\text{RHL} = \lim_{k\to 0^+} f(x) = \lim_{k\to 0^+}(\sin x - \cos x) = 0 - 1 = -1$$

Hence, at $x \neq 0, \text{LHL} = \text{RHL} = f(x)$

Hence, the function f is continuous at $x \neq 0$.

Second case: If, $k = 0, f(k) = -1$

and $\lim_{x\to k} f(x) = \lim_{x\to k}(-1) = -1$ Here, $\lim_{x\to k} f(x) = f(k)$

Hence, the function f is continuous at $x = 0$

Hence, the function f is continuous for all real numbers.

26. Find the values of k so that the function f is continuous at the indicated point.

$$f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, & \text{If } x \neq \frac{\pi}{2} \\ 3, & \text{If } x = \frac{\pi}{2} \end{cases} \text{ at } x = \frac{\pi}{2}$$

Solution:

Given function is defined by $f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, & \text{If } x \neq \frac{\pi}{2} \\ 3, & \text{If } x = \frac{\pi}{2} \end{cases}$ at $x = \frac{\pi}{2}$

Given that the function is continuous at $x = \frac{\pi}{2}$. Therefore, $\text{LHL} = \text{RHL} = f\left(\frac{\pi}{2}\right)$

$$\Rightarrow \lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^+} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \lim_{x \to \frac{\pi}{2}^-} \frac{k\cos x}{\pi - 2x} = \lim_{x \to \frac{\pi}{2}^+} \frac{k\cos x}{\pi - 2x} = 3$$

$$\Rightarrow \lim_{h \to 0} \frac{k\cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)} = \lim_{h \to 0} \frac{k\cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)} = 3$$

$$\Rightarrow \lim_{h \to 0} \frac{k\sin h}{2h} = \lim_{h \to 0} \frac{-k\sin h}{-2h} = 3$$

$$\Rightarrow \frac{k}{2} = \frac{k}{2} = 3 \quad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

$$\Rightarrow k = 6$$

Hence, for $k = 6$, the given function f is continuous at the indicated point.

27. Find the values of k so that the function f is continuous at the indicated point.

$$f(x) = \begin{cases} kx^2, & \text{If } x \leq 2 \\ 3, & \text{If } x > 2 \end{cases} \text{ at } x = 2$$

Solution:

Given function is defined by $f(x) = \begin{cases} kx^2, & \text{If } x \leq 2 \\ 3, & \text{If } x > 2 \end{cases}$ at $x = 2$

Given that the function is continuous at $x = 2$.

Therefore, LHL $=$ RHL $= f(2)$

$\Rightarrow \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$

$\Rightarrow \lim_{x \to 2^-} kx^2 = \lim_{x \to 2^+} 3 = k(2)^2$

$\Rightarrow 4k = 3 = 4k$

$\Rightarrow k = \frac{3}{4}$

Hence, for $k = \frac{3}{4}$, the given function f is continuous at the indicated point.

28. Find the values of k so that the function f is continuous at the indicated point.

$$f(x) = \begin{cases} kx + 1, & \text{If } x \le \pi \\ \cos x, & \text{If } x > \pi \end{cases} \text{ at } x = \pi$$

Solution:

Given function is $f(x) = \begin{cases} kx + 1, & \text{If } x \le \pi \\ \cos x, & \text{If } x > \pi \end{cases}$ at $x = \pi$

Given that the function is continuous at $x = \pi$,

Therefore, LHL $=$ RHL $= f(\pi)$

$\Rightarrow \lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x) = f(\pi)$

$\Rightarrow \lim_{x \to \pi^-} kx + 1 = \lim_{x \to \pi^+} \cos x = k(\pi) + 1$

$\Rightarrow k(\pi) + 1 = \cos \pi = k\pi + 1$

$\Rightarrow k\pi + 1 = -1 = k\pi + 1$

$\Rightarrow \pi k = -2$

$\Rightarrow k = -\frac{2}{\pi}$

Hence, for $k = -\frac{2}{\pi}$, the given function f is continuous at the indicated point.

29. Find the values of k so that the function f is continuous at the indicated point.

$$f(x) = \begin{cases} kx + 1, & \text{If } x \le 5 \\ 3x - 5, & \text{If } x > 5 \end{cases} \text{ at } x = 5$$

Solution:

Given function is defined by $f(x) = \begin{cases} kx+1, & \text{If } x \le 5 \\ 3x-5, & \text{If } x > 5 \end{cases}$ at $x = 5$

Given that the function is continuous at $x = 5$.

Therefore, $\text{LHL} = \text{RHL} = f(5)$

$\Rightarrow \lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = f(5)$

$\Rightarrow \lim_{x\to 5^-} kx+1 = \lim_{x\to 5^+} 3x-5 = 5k+1$

$\Rightarrow 5k+1 = 15-5 = 5k+1$

$\Rightarrow 5k = 9$

$\Rightarrow k = \frac{9}{5}$

Hence, for $k = \frac{9}{5}$, the given function f is continuous at the indicated point.

30. Find the values of a and b such that the function defined by

$$f(x) = \begin{cases} 5, & \text{If } x \le 2 \\ ax+b, & \text{If } 2 < x < 10 \\ 21, & \text{If } x \ge 10 \end{cases}$$

is continuous function.

Solution:

Given function is $f(x) = \begin{cases} 5, & \text{If } x \le 2 \\ ax+b, & \text{If } 2 < x < 10 \\ 21, & \text{If } x \ge 10 \end{cases}$

Given that the function is continuous at $x = 2$. Therefore, $\text{LHL} = \text{RHL} = f(2)$

$\Rightarrow \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$

$\Rightarrow \lim_{x\to 2^-} 5 = \lim_{x\to 2^+} ax+b = 5$

$\Rightarrow 2a+b = 5$ ...(i)

Given that the function is continuous at $x = 10$. Therefore, $\text{LHL} = \text{RHL} = f(10)$

$\Rightarrow \lim_{x\to 10^-} f(x) = \lim_{x\to 10^+} f(x) = f(10)$

$\Rightarrow \lim_{x\to 10^-} ax+b = \lim_{x\to 10^+} 21 = 21$

$\Rightarrow 10a+b = 21$ ...(ii)

Solving the equation (i) and (ii), we get

$a = 2 \quad b = 1$

31. Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

Solution:

Given function is defined by $f(x) = \cos(x^2)$

Assuming that the functions are well defined for all real numbers, we can write the given function f in the combination of g and $h(f = goh)$. Where, $g(x) = \cos x$ and $h(x) = x^2$, if g and h both are continuous function then f also be continuous.

$[\because goh(x) = g(h(x)) = g(x^2) = \cos(x^2)]$

Let the function $g(x)$ be $\cos x$

Let k be any real number. At $x = k, g(k) = \cos k$

$\lim_{x\to k} g(x) = \lim_{x\to k} \cos x = \lim_{h\to 0} \cos(k+h) = \lim_{h\to 0} \cos k \cos h - \sin k \sin h = \cos k$

Here, $\lim_{x\to k} g(x) = g(k)$. Hence, t\he function g is continuous for all real numbers.

And let the function $h(x)$ be x^2

Let k be any real number. At $x = k, h(k) = k^2$

$\lim_{x\to k} h(x) = \lim_{x\to k} x^2 = k^2$

Here, $\lim_{x\to k} h(x) = h(k)$. Hence, the function h is continuous for all real numbers.

Therefore, g and h both are continuous function. Hence, f is continuous.

32. Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

Solution:

Given that the function is defined by $f(x) = |\cos x|$

Assuming that the functions are well defined for all real numbers, we can write the given function f in the combination of g and $h(f = goh)$. Where, $g(x) = |x|$ and $h(x) = \cos x$. If g and h both are continuous function then f also be continuous.

$[\because goh(x) = g(h(x)) = g(\cos x) = |\cos x|]$

Let the function $g(x)$ be $|x|$

Rearranging the function g, we get

$$g(x) = \begin{cases} -x, & \text{If } x < 0 \\ x, & \text{If } x \geq 0 \end{cases}$$

Let k be any real number. According to question, $k < 0$ or $k = 0$ or $k > 0$

First case: If $k < 0$,

$g(k) = 0$ and $\lim_{x \to k} g(x) = \lim_{x \to k}(-x) = 0$, here, $\lim_{x \to k} g(x) = g(k)$

Hence, the function g is continuous for all real numbers less than 0.

Second case: If $k = 0, g(0) = 0 + 1 = 1$

$\text{LHL} = \lim_{x \to 0^-} g(x) = \lim_{x \to 0^-}(-x) = 0$

$\text{RHL} = \lim_{x \to 0^+} g(x) = \lim_{x \to 0^+}(x) = 0,$

Here at $x = 0, \text{LHL} = \text{RHL} = g(0)$

Hence, the function g is continuous at $x = 0$.

Third case: If $k > 0$,

$g(k) = 0$ and $\lim_{x \to k} g(x) = \lim_{x \to k}(x) = 0$. Here, $\lim_{x \to k} g(x) = g(k)$

Hence, the function g is continuous for all real numbers greater than 0.

Hence, the function g is continuous for all real numbers.

And let the function $h(x)$ be $\cos x$

Let k be any real number. At $x = k, h(k) = \cos k$

$\lim_{x \to k} h(x) = \lim_{x \to k} \cos x = \cos k$

Here, $\lim_{x \to k} h(x) = h(k)$. Hence, the function h is continuous for all real numbers.

Therefore, g and h both are continuous function. Hence, f is continuous.

33. Examine that $\sin|x|$ is a continuous function.

Solution:

Let the given function be $f(x) = \sin|x|$

Assuming that the functions are well defined for all real numbers, we can write the given function f in the combination of g and $h(f = hog)$. Where, $h(x) = \sin x$ and $g(x) = |x|$. If g and h both are continuous function then f also be continuous.

$[\because hog(x) = h(g(x)) = h(|x|) = \sin|x|]$

Function $h(x) = \sin x$

Let k be any real number. At $x = k, h(k) = \sin k$

$\lim_{x \to k} h(x) = \lim_{x \to k} \sin x = \sin k$

Here, $\lim_{x \to k} h(x) = h(k)$. Hence, the function h is continuous for all real numbers.

Function $g(x) = |x|$

Redefining the function g, we get

$$g(x) = \begin{cases} -x, & \text{If } x < 0 \\ x, & \text{If } x \geq 0 \end{cases}$$

Let k be any real number. According to question, $k < 0$ or $k = 0$ or $k > 0$

First case: If $k < 0$,

$g(k) = 0$ and $\lim_{x \to k} g(x) = \lim_{x \to k}(-x) = 0$. Here, $\lim_{x \to k} g(x) = g(k)$

Hence, the function g is continuous for all real numbers less than 0.

Second case: If $k = 0, g(0) = 0 + 1 = 1$

$\text{LHL} = \lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-x) = 0$

$\text{RHL} = \lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x) = 0$

Here at $x = 0, \text{LHL} = \text{RHL} = g(0)$

Hence at $x = 0$, the function g is continuous.

Third case: If $k > 0$,

$g(k) = 0$ and $\lim_{x \to k} g(x) = \lim_{x \to k}(x) = 0$. Here, $\lim_{x \to k} g(x) = g(k)$

Hence, the function g is continuous for all real numbers greater than 0.

Hence, the function g is continuous for all real numbers.

Therefore, g and h both are continuous function. Hence, f is continuous.

34. Find all the points of discontinuity of f defined by $f(x) = |x| - |x + 1|$.

Solution:

Given that the function is defined by $f(x) = |x| - |x + 1|$

Assuming that the functions are well defined for all real numbers, we can write the given function f in the combination of g and $h(f = g - h)$, where, $g(x) = |x|$ and $h(x) = |x + 1|$. If g and h both are continuous function then f also be continuous.

Function $g(x) = |x|$

Redefining the function g, we get,

$$g(x) = \begin{cases} -x, & \text{If } x < 0 \\ x, & \text{If } x \geq 0 \end{cases}$$

Let k be any real number. According to question, $k < 0$ or $k = 0$ or $k > 0$

First case: If $k < 0$,

$g(k) = 0$ and $\lim_{x \to k} g(x) = \lim_{x \to k}(-x) = 0$. Here, $\lim_{x \to k} g(x) = g(k)$

Hence, the function g is continuous for all real numbers less than 0.

Second case: If $k = 0, g(0) = 0 + 1 = 1$

$\text{LHL} = \lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-x) = 0$ and $\text{RHL} = \lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x) = 0$

Here, at $x = 0, \text{LHL} = \text{RHL} = \text{g}(0)$

Hence, the function g is continuous at $x = 0$.

Third case: If $k > 0$,

$g(k) = 0$ and $\lim_{x \to k} g(x) = \lim_{x \to k}(x) = 0$. Here, $\lim_{x \to k} g(x) = g(k)$

Hence, the function g is continuous for all real numbers more than 0.

Hence, the function g is continuous for all real numbers.

Function $h(x) = |x + 1|$

Redefining the function h, we get

$$h(x) = \begin{cases} -(x+1), & \text{If } x < -1 \\ x + 1, & \text{If } x \geq -1 \end{cases}$$

Let k be any real number. According to question, $k < -1$ or $k = -1$ or $k > -1$

First case: If $k < -1$,

$h(k) = -(k + 1)$ and $\lim_{x \to k} h(x) = \lim_{x \to k} -(k + 1) = -(k + 1)$. Here, $\lim_{x \to k} h(x) = h(k)$

Hence, the function g is continuous for all real numbers less than -1.

Second case: If $k = -1, h(-1) = -1 + 1 = 0$

$\text{LHL} = \lim_{x \to -1^-} h(x) = \lim_{x \to -1^-} -(-1 + 1) = 0$

$\text{RHL} = \lim_{x \to -1^+} h(x) = \lim_{x \to -1^+} (x + 1) = -1 + 1 = 0$

Here at $x = -1, \text{LHL} = \text{RHL} = h(-1)$

Hence, the function h is continuous at $x = -1$.

Third case: If $k > -1$

$h(k) = k + 1$ and $\lim_{x \to k} h(x) = \lim_{x \to k}(k + 1) = k + 1$. Here, $\lim_{x \to k} h(x) = h(k)$

Hence, the function g is continuous for all real numbers greater than -1.

Hence, the function h is continuous for all real numbers.

Therefore, g and h both are continuous function. Hence, f is continuous.

Exercise 5.2

1. Differentiate the functions with respect to x

$\sin(x^2+5)$

Solution:

Let $y=\sin(x^2+5)$

Therefore,

$\frac{dy}{dx}=\cos(x^2+5).\frac{d}{dx}(x^2+5)$

$=\cos(x^2+5).2x$

Hence, $\frac{d(\sin(x^2+5))}{dx} = \cos(x^2+5).2x$

2. Differentiate the functions with respect to x

$\cos(\sin x)$

Solution:

Let $y=\cos(\sin x)$

Therefore,

$\frac{dy}{dx}=-\sin(\sin x).\frac{d}{dx}(\sin x)$

$=-\sin(\sin x).\cos x$

Hence, $\frac{d((\cos(\sin x)))}{dx} = -\sin(\sin x).\cos x.$

3. Differentiate the functions with respect to x

$\sin(ax+b)$

Solution:

Let $y=\sin(ax+b)$

Therefore,

$\frac{dy}{dx}=\cos(ax+b).\frac{d}{dx}(ax+b)$

$= \cos(ax+b).a$

Hence, $\frac{d(\sin(ax+b))}{dx} = \cos(ax+b).a$

4. Differentiate the functions with respect to x

$\sec(\tan(\sqrt{x}))$

Solution:

Let $y = \sec(\tan(\sqrt{x}))$

Therefore,

$\frac{dy}{dx} = \sec(\tan\sqrt{x})\tan(\tan\sqrt{x}).\frac{d}{dx}(\tan\sqrt{x})$

$= \sec(\tan\sqrt{x})\tan(\tan\sqrt{x}).\sec^2\sqrt{x}\frac{d}{dx}(\sqrt{x})$

$= \sec(\tan\sqrt{x})\tan(\tan\sqrt{x}).\sec^2\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)$

Hence, $\frac{d(\sec(\tan(\sqrt{x})))}{dx} = \sec(\tan\sqrt{x})\tan(\tan\sqrt{x}).\sec^2\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)$

5. Differentiate the functions with respect to x

$\frac{\sin(ax+b)}{\cos(cx+d)}$

Solution:

Let $y = \frac{\sin(ax+b)}{\cos(cx+d)}$

Therefore,

$$\frac{dy}{dx} = \frac{\cos(cx+d).\frac{d}{dx}\sin(ax+b) - \sin(ax+b).\frac{d}{dx}\cos(cx+d)}{[\cos(cx+d)]^2}$$

$$= \frac{\cos(cx+d).\sin(ax+b).\frac{d}{dx}(ax+b) - \sin(ax+b).\left[-\sin(cx+d).\frac{d}{dx}(cx+d)\right]}{\cos^2(cx+d)}$$

$$= \frac{\cos(cx+d).\sin(ax+b).a + \sin(ax+b).\sin(cx+d)c}{\cos^2(cx+d)}$$

Hence, $\frac{d\left(\frac{\sin(ax+b)}{\cos(cx+d)}\right)}{dx} = \frac{\cos(cx+d).\sin(ax+b).a+\sin(ax+b).\sin(cx+d)c}{\cos^2(cx+d)}$

6. Differentiate the functions with respect to x

$\cos x^3 . \sin^2(x^5)$

Solution:

Let $y = \cos x^3 . \sin^2(x^5)$

Therefore,

$\frac{dy}{dx} = \cos x^3 . \frac{d}{dx} \sin^2(x^5) + \sin^2(x^5) . \frac{d}{dx} \cos x^3$

$= \cos x^3 . 2 \sin x^5 \cos x^5 . \frac{d}{dx} x^5 + \sin^2(x^5)[-\sin x^3] . \frac{d}{dx} x^3$

$= \cos x^3 . 2 \sin x^5 \cos x^5 . 5x^4 - \sin^2(x^5) \sin x^3 . 3x^2$

Hence, $\frac{d(\cos x^3 . \sin^2(x^5))}{dx} = \cos x^3 . 2 \sin x^5 \cos x^5 . 5x^4 - \sin^2(x^5) \sin x^3 . 3x^2$

7. Differentiate the functions with respect to x

$2\sqrt{\cot(x^2)}$

Solution:

Let $y = 2\sqrt{\cot(x^2)}$

Therefore,

$\frac{dy}{dx} = 2 . \frac{1}{2\sqrt{\cot(x^2)}} . \frac{d}{dx}[\cot(x^2)]$

$= \frac{1}{\sqrt{\cot(x^2)}} . [-\operatorname{cossec} x^2] . \frac{d}{dx} x^2$

$= \frac{1}{\sqrt{\cot(x^2)}} . [-\operatorname{cossec} x^2] . 2x$

Hence, $\frac{d\left(2\sqrt{\cot(x^2)}\right)}{dx} = \frac{1}{\sqrt{\cot(x^2)}} . [-\operatorname{cossec} x^2] . 2x$

8. Differentiate the functions with respect to x

$\cos(\sqrt{x})$

Solution:

Let $y = \cos(\sqrt{x})$

Therefore,

$\frac{dy}{dx} = -\sin(\sqrt{x}).\frac{d}{dx}\sqrt{x}$

$= -\sin(\sqrt{x}).\frac{1}{2\sqrt{x}}$

Hence, $\frac{d(\cos(\sqrt{x}))}{dx} = -\sin(\sqrt{x}).\frac{1}{2\sqrt{x}}$

9. Prove that the function f given by $f(x) = |x-1|, x \in R$, is not differentiable at $x = 1$.

Solution:

At $x = 1$,

$$\text{LHD} = \lim_{h\to0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to0}\frac{|1-h-1|-|1-1|}{-h} = \lim_{h\to0}\frac{h}{-h} = -1$$

$$\text{RHD} = \lim_{h\to0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to0}\frac{|1+h-1|-|1-1|}{h} = \lim_{h\to0}\frac{h}{h} = 1$$

Here, $\text{LHD} \neq \text{RHD}$, therefore,

the function $f(x) = |x-1|, x \in R$, is not differentiable at $x = 1$.

10. Prove that the greatest integer function defined by $f(x) = [x], 0 < x < 3$, is not differentiable at $x = 1$ and $x = 2$.

Solution:

At $x = 1$,

$$\text{LHD} = \lim_{h\to0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to0}\frac{[1-h]-[1]}{-h} = \lim_{h\to0}\frac{0-1}{-h} = \infty$$

$$\text{RHD} = \lim_{h\to0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to0}\frac{[1+h]-[1]}{h} = \lim_{h\to0}\frac{1-1}{h} = 0$$

Here, $\text{LHD} \neq \text{RHD}$, therefore,

The function $f(x) = [x], 0 < x < 3$, is not differentiable at $x = 1$.

At $x = 2$,

$$\text{LHD} = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{[2-h]-[2]}{-h} = \lim_{h\to 0}\frac{1-2}{-h} = \infty$$

$$\text{RHD} = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{[2+h]-[2]}{h} = \lim_{h\to 0}\frac{2-2}{h} = 0$$

Here, LHD $\neq$ RHD, therefore,

The function $f(x) = [x], 0 < x < 3$, is not differentiable at $x = 2$.

Exercise 5.3

Find $\frac{dy}{dx}$ in the following:

1. $2x + 3y = \sin x$

Solution:

Given equation is $2x + 3y = \sin x$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}\sin x$$

$$\Rightarrow 2 + 3\frac{dy}{dx} = \cos x$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x - 2}{3}$$

2. $2x + 3y = \sin y$

Solution:

Given equation is $2x + 3y = \sin y$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}\sin y \quad \Rightarrow 2 + 3\frac{dy}{dx} = \cos y\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(\cos y - 3) = 2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\cos y - 3}$$

3. $ax + by^2 = \cos y$

Solution:

Given equation is $ax + by^2 = \cos y$

Differentiating both sides with respect to x, we get

$\frac{d}{dx}(ax) + \frac{d}{dx}(by^2) = \frac{d}{dx}\cos y \quad \Rightarrow a + 2by\frac{dy}{dx} = -\sin y\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}(2by + \sin y) = -a \quad \Rightarrow \frac{dy}{dx} = -\frac{a}{2by+\sin y}$

4. $xy + y^2 = \tan x + y$

Solution:

Given equation is $xy + y^2 = \tan x + y$

Differentiating both sides with respect to x, we get

$\frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}\tan x + \frac{dy}{dx}$

$\Rightarrow x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}(x + 2y - 1) = \sec^2 x - y$

$\Rightarrow \frac{dy}{dx} = \frac{\sec^2 x - y}{x+2y-1}$

5. $x^2 + xy + y^2 = 100$

Solution:

Given equation is $x^2 + xy + y^2 = 100$

Differentiating both sides with respect to x, we get

$\frac{d}{dx}x^2 + \frac{d}{dx}(xy) + \frac{d}{dx}y^2 = \frac{d}{dx}(100)$

$\Rightarrow 2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx}(x + 2y) = 2x + y \quad \Rightarrow \frac{dy}{dx} = \frac{2x+y}{x+2y}$

6. $x^3 + x^2y + xy^2 + y^3 = 81$

Solution:

Given equation is $x^3 + x^2y + xy^2 + y^3 = 81$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}x^3 + \frac{d}{dx}(x^2y) + \frac{d}{dx}(xy^2) + \frac{d}{dx}y^3 = \frac{d}{dx}81$$

$$\Rightarrow 3x^2 + x^2\frac{dy}{dx} + y.2x + x.2y\frac{dy}{dx} + y^2.1 + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(x^2 + 2xy + 3y^2) = -(3x^2 + 2xy + y^2) \quad \Rightarrow \frac{dy}{dx} = -\frac{3x^2+2xy+y^2}{x^2+2xy+3y^2}$$

7. $\sin^2 y + \cos xy = k$

Solution:

Given equation is $\sin^2 y + \cos xy = k$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}\sin^2 y + \frac{d}{dx}\cos xy = \frac{d}{dx}k$$

$$\Rightarrow 2\sin y\cos y\frac{dy}{dx} - \sin xy\left(x\frac{dy}{dx} + y\right) = 0$$

$$\Rightarrow \sin 2y\frac{dy}{dx} - x\sin xy\frac{dy}{dx} - y\sin xy = 0$$

$$\Rightarrow (\sin 2y - x\sin xy)\frac{dy}{dx} = y\sin xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

8. $\sin^2 x + \cos^2 y = 1$

Solution:

Given equation is $\sin^2 x + \cos^2 y = 1$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}\sin^2 x + \frac{d}{dx}\cos^2 y = \frac{d}{dx}1$$

$$\Rightarrow 2\sin x\cos x + 2\cos y\,(-\sin y)\frac{dy}{dx} = 0$$

$\Rightarrow \sin 2x - \sin 2y \frac{dy}{dx} = 0 \quad \Rightarrow \frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}$

9. $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Solution:

Given equation is $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Let $x = \tan\theta$

Therefore, $y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}x$

$\Rightarrow y = 2\tan^{-1}x$

Differentiating both sides with respect to x, we get

$\frac{dy}{dx} = \frac{2}{1+x^2}$

10. $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

Solution:

Given equation is $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

Let $x = \tan\theta$

Therefore, $y = \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$

$= \tan^{-1}(\tan 3\theta) = 3\theta = 3\tan^{-1}x$

$\Rightarrow y = 3\tan^{-1}x$

Differentiating both sides with respect to x, we get

$\frac{dy}{dx} = \frac{3}{1+x^2}$

11. $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

Solution:

Given equation is $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Let $x = \tan\theta$

Therefore, $y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= \cos^{-1}(\cos 2\theta) = 2\theta = 2\tan^{-1} x$

$\Rightarrow y = 2\tan^{-1} x$

Differentiating both sides with respect to x, we get

$\frac{dy}{dx} = \frac{2}{1+x^2}$

12. $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

Solution:

Given equation is $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Let $x = \tan\theta$

Therefore,

$y = \sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= \sin^{-1}(\cos 2\theta) = \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - 2\theta\right)\right\} = \frac{\pi}{2} - 2\theta = \frac{\pi}{2} - 2\tan^{-1} x$

$\Rightarrow y = \frac{\pi}{2} - 2\tan^{-1} x$

Differentiating both sides with respect to x, we get

$\frac{dy}{dx} = 0 - \frac{2}{1+x^2} = -\frac{2}{1+x^2}$

13. $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right), -1 < x < 1$

Solution:

Given equation is $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$

Let $x = \tan\theta$

Therefore, $y = \cos^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$= \cos^{-1}(\sin 2\theta) = \cos^{-1}\left\{\cos\left(\frac{\pi}{2} - 2\theta\right)\right\} = \frac{\pi}{2} - 2\theta = \frac{\pi}{2} - 2\tan^{-1} x$

$\Rightarrow y = \frac{\pi}{2} - 2\tan^{-1} x$

Differentiating both sides with respect to x, we get

$\frac{dy}{dx} = 0 - \frac{2}{1+x^2} = -\frac{2}{1+x^2}$

14. $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

Solution:

Given equation is $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

Let $x = \sin\theta$

Therefore, $y = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$= \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\sin^{-1} x$

$\Rightarrow y = 2\sin^{-1} x$

Differentiating both sides with respect to x, we get

$\frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$

15. $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$

Solution:

Given equation is $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$

Let $x = \cos\theta$

Therefore, $y = \sec^{-1}\left(\frac{1}{2\cos^2\theta-1}\right) = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$

$\sec^{-1}(\sec 2\theta) = 2\theta = 2\cos^{-1} x$

$\Rightarrow y = 2\cos^{-1} x$

Differentiating both sides with respect to x, we get

$\frac{dy}{dx} = -\frac{2}{\sqrt{1-x^2}}$

Exercise 5.4

1. Differentiate the following w.r.t. x:

$\frac{e^x}{\sin x}$

Solution:

Given expression is $\frac{e^x}{\sin x}$

Let $y = \frac{e^x}{\sin x}$ therefore,

$\frac{dy}{dx} = \frac{e^x.\frac{d}{dx}\sin x - \sin x\frac{d}{dx}e^x}{\sin^2 x} = \frac{e^x.\cos x - \sin x.e^x}{\sin^2 x} = \frac{e^x(\cos x - \sin x)}{\sin^2 x}$

2. $e^{\sin^{-1} x}$

Solution:

Given expression is $e^{\sin^{-1} x}$

Let $y = e^{\sin^{-1} x}$, therefore,

$\frac{dy}{dx} = e^{\sin^{-1} x}.\frac{d}{dx}\sin^{-1} x = e^{\sin^{-1} x}.\frac{1}{\sqrt{1-x^2}} = \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}.$

3. e^{x^3}

Solution:

Given expression is e^{x^3}

Let $y = e^{x^3}$, therefore,

$\frac{dy}{dx} = e^{x^3}.\frac{d}{dx}x^3 = e^{x^3}.3x^2 = 3x^2e^{x^3}$

4. $\sin(\tan^{-1} e^{-x})$

Solution:

Given expression is $\sin(\tan^{-1} e^{-x})$

Let $y = \sin(\tan^{-1} e^{-x})$, therefore,

$\frac{dy}{dx} = \cos(\tan^{-1} e^{-x}) . \frac{d}{dx} \tan^{-1} e^{-x} = \cos(\tan^{-1} e^{-x}) . \frac{1}{1+(e^{-x})^2} . \frac{d}{dx} e^{-x}$

$= \cos(\tan^{-1} e^{-x}) . \frac{1}{1+e^{-2x}} . (-e^{-x}) = -\frac{e^{-x}\cos(\tan^{-1} e^{-x})}{1+e^{-2x}}.$

5. $\log(\cos e^x)$

Solution:

Given expression is $\log(\cos e^x)$

Let $y = \log(\cos e^x)$,

Therefore,

$\frac{dy}{dx} = \frac{1}{\cos e^x} . \frac{d}{dx} \cos e^x = \frac{1}{\cos e^x} (-\sin e^x) \frac{d}{dx} e^x = -\tan e^x . e^x$

6. $e^x + e^{x^2} + \cdots + e^{x^5}$

Solution:

Given expression is $e^x + e^{x^2} + \cdots + e^{x^5}$

Let $y = e^x + e^{x^2} + e^{x^3} + e^{x^4} + e^{x^5}$, therefore,

$\frac{dy}{dx} = e^x + e^{x^2} \frac{d}{dx} x^2 + e^{x^3} \frac{d}{dx} x^3 + e^{x^4} \frac{d}{dx} x^4 + e^{x^5} \frac{d}{dx} x^5$

$= e^x + e^{x^2} . 2x + e^{x^3} . 3x^2 + e^{x^4} . 4x^3 + e^{x^5} . 5x^4$

$= e^x + 2xe^{x^2} + 3x^2 e^{x^3} + 4x^3 e^{x^4} + 5x^4 e^{x^5}$

7. $\sqrt{e^{\sqrt{x}}}, x > 0$

Solution:

Given expression is $\sqrt{e^{\sqrt{x}}}, x > 0$

Let $y = \sqrt{e^{\sqrt{x}}}$

Therefore,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{e^{\sqrt{x}}}} \frac{d}{dx} e^{\sqrt{x}} = \frac{1}{2\sqrt{e^{\sqrt{x}}}} . e^{\sqrt{x}} . \frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{e^{\sqrt{x}}}} . e^{\sqrt{x}} . \frac{1}{2\sqrt{x}} = \frac{\sqrt{e^{\sqrt{x}}}}{4\sqrt{x}}$$

8. $\log(\log x), x > 1$

Solution:

Given expression is $\log(\log x), x > 1$

Let $y = \frac{e^x}{\sin x}$

Therefore,

$$\frac{dy}{dx} = \frac{1}{\log x} . \frac{d}{dx} \log x = \frac{1}{\log x} . \frac{1}{x} = \frac{1}{x \log x}$$

9. $\frac{\cos x}{\log x}, x > 0$

Solution:

Given expression is $\frac{\cos x}{\log x}, x > 0$

Let $y = \frac{\cos x}{\log x}$

Therefore, $\frac{dy}{dx} = \frac{\log x \frac{d}{dx} \cos x - \cos x \frac{d}{dx} \log x}{(\log x)^2} = \frac{\log x . (-\sin x) - \cos x . \frac{1}{x}}{(\log x)^2} = \frac{-(x \sin x \log x + \cos x)}{x (\log x)^2}$

10. $\cos(\log x + e^x)$

Solution:

Given expression is $\cos(\log x + e^x)$

Let $y = \cos(\log x + e^x)$

Therefore,

$$\frac{dy}{dx} = -\sin(\log x + e^x).\frac{d}{dx}(\log x + e^x) = -\sin(\log x + e^x).\left(\frac{1}{x} + e^x\right)$$

Exercise 5.5

1. Differentiate the functions given

$\cos x.\cos 2x.\cos 3x$

Solution:

Given function is $\cos x.\cos 2x.\cos 3x$

Let $y = \cos x.\cos 2x.\cos 3x$, taking log on both the sides

$\log y = \log\cos x + \log\cos 2x + \log\cos 3x$

Therefore,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\cos x}.\frac{d}{dx}\cos x + \frac{1}{\cos 2x}.\frac{d}{dx}\cos 2x + \frac{1}{\cos 3x}.\frac{d}{dx}\cos 3x$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{\cos x}.(-\sin x) + \frac{1}{\cos 2x}.(-\sin 2x).2 + \frac{1}{\cos 3x}.(-\sin 3x).3\right]$$

$$\Rightarrow \frac{dy}{dx} = \cos x.\cos 2x.\cos 3x\,[-\tan x - 2\tan 2x - 3\tan 3x]$$

2. Differentiate the functions given

$$\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Solution:

Given function is $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Let $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$, taking log on both the sides

$$\log y = \frac{1}{2}[\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)]$$

Therefore,

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{(x-1)}+\frac{1}{(x-2)}-\frac{1}{(x-3)}-\frac{1}{(x-4)}-\frac{1}{(x-5)}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{(x-1)}+\frac{1}{(x-2)}-\frac{1}{(x-3)}-\frac{1}{(x-4)}-\frac{1}{(x-5)}\right]$$

3. Differentiate the functions given

$(\log x)^{\cos x}$

Solution:

Given function is $(\log x)^{\cos x}$

Let $y=(\log x)^{\cos x}$, taking log on both the sides

$\log y=\log(\log x)^{\cos x}=\cos x.\log\log x$

Therefore,

$$\frac{1}{y}\frac{dy}{dx}=\cos x.\frac{d}{dx}\log\log x+\log\log x.\frac{d}{dx}\cos x$$

$$\Rightarrow \frac{dy}{dx}=y\left[\cos x.\frac{1}{\log x}.\frac{1}{x}+\log\log x.(-\sin x)\right]$$

$$\Rightarrow \frac{dy}{dx}=(\log x)^{\cos x}\left[\frac{\cos x-\sin x\log\log x}{x\log x}\right]$$

4. Differentiate the functions given

$x^x-2^{\sin x}$

Solution:

Given function is $x^x-2^{\sin x}$

Let $u=x^x$ and $v=2^{\sin x}$ therefore, $y=u-v$

Differentiating with respect to x on both sides

$$\frac{dy}{dx}=\frac{du}{dx}-\frac{dv}{dx}\quad ...(i)$$

Here, $u=x^x$, taking log on both the sides

$\log u = x \log x$, therefore,

$\frac{1}{u}\frac{du}{dx} = x.\frac{d}{dx}\log x + \log x.\frac{d}{dx}x = x.\frac{1}{x} + \log x.1 = 1 + \log x$

$\frac{du}{dx} = u[1 + \log x] = x^x[1 + \log x]$ ...(ii)

and $v = 2^{\sin x}$, taking log on both the sides

$\log v = \sin x \log 2$, therefore,

$\frac{1}{v}\frac{dv}{dx} = \log 2 \cdot \frac{d}{dx}\sin x = \log 2 \cdot \cos x$

$\frac{dv}{dx} = v[\cos x \log 2] = 2^{\sin x}[\cos x \log 2]$ ...(iii)

Putting the value of $\frac{du}{dx}$ from (ii) and $\frac{dv}{dx}$ from (iii) in equation (i), we get

$\frac{dy}{dx} = x^x[1 + \log x] - 2^{\sin x}[\cos x \log 2]$

5. Differentiate the functions given

$(x + 3)^2.(x + 4)^3.(x + 5)^4$

Solution:

Given function is $(x + 3)^2.(x + 4)^3.(x + 5)^4$

Let $y = (x + 3)^2.(x + 4)^3.(x + 5)^4$, taking log on both the sides

$\log y = 2\log(x + 3) + 3\log(x + 4) + 4\log(x + 5)$

Therefore,

$\frac{1}{y}\frac{dy}{dx} = 2.\frac{1}{(x+3)} + 3.\frac{1}{(x+4)} + 4.\frac{1}{(x+5)}$

$\Rightarrow \frac{dy}{dx} = y\left[\frac{2(x + 4)(x + 5) + 3(x + 3)(x + 5) + 4(x + 3)(x + 4)}{(x + 3)(x + 4)(x + 5)}\right]$

$\Rightarrow \frac{dy}{dx} = y\left[\frac{2(x^2 + 9x + 20) + 3(x^2 + 8x + 15) + 4(x^2 + 7x + 12)}{(x + 3)(x + 4)(x + 5)}\right]$

$\Rightarrow \frac{dy}{dx} = (x + 3)^2 \cdot (x + 4)^3 \cdot (x + 5)^4\left[\frac{9x^2+70x+133}{(x+3)(x+4)(x+5)}\right]$

$\Rightarrow \frac{dy}{dx} = (x + 3) \cdot (x + 4)^2 \cdot (x + 5)^3(9x^2 + 70x + 133)$

6. Differentiate the functions given

$\left(x + \frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

Solution:

Given function is $\left(x+\frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

Let $u = \left(x+\frac{1}{x}\right)^x$ and $v = x^{\left(1+\frac{1}{x}\right)}$, therefore, $y = u + v$

Differentiating with respect to x, we get

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...(i)

Here, $u = \left(x+\frac{1}{x}\right)^x$, taking log on both the sides

$\log u = x \log\left(x+\frac{1}{x}\right)$, therefore,

$\frac{1}{u}\frac{du}{dx} = x \cdot \frac{d}{dx}\log\left(x+\frac{1}{x}\right) + \log\left(x+\frac{1}{x}\right) \cdot \frac{d}{dx}x$

$= x \cdot \frac{1}{\left(x+\frac{1}{x}\right)} \cdot \left(1-\frac{1}{x^2}\right) + \log\left(x+\frac{1}{x}\right) \cdot 1 = \frac{x^2}{x^2+1} \cdot \frac{x^2-1}{x^2} + \log\left(x+\frac{1}{x}\right)$

$\frac{du}{dx} = \left(x+\frac{1}{x}\right)^x \left[\frac{x^2-1}{x^2+1} + \log\left(x+\frac{1}{x}\right)\right]$ ...(ii)

and $v = x^{\left(1+\frac{1}{x}\right)}$, taking log on both the sides

$\log v = \left(1+\frac{1}{x}\right)\log x$, therefore,

$\frac{1}{v}\frac{dv}{dx} = \left(1+\frac{1}{x}\right) \cdot \frac{d}{dx}\log x + \log x \cdot \frac{d}{dx}\left(1+\frac{1}{x}\right) = \left(1+\frac{1}{x}\right) \cdot \frac{1}{x} + \log x \cdot \left(-\frac{1}{x^2}\right)$

$\frac{dv}{dx} = v\left[\left(\frac{x^2+1}{x}\right) \cdot \frac{1}{x} - \frac{\log x}{x^2}\right] = x^{\left(1+\frac{1}{x}\right)}\left[\frac{x^2+1-\log x}{x^2}\right]$ ...(iii)

Putting the value of $\frac{du}{dx}$ from (ii) and $\frac{dv}{dx}$ from (iii) in equation (i), we get

$\frac{dy}{dx} = \left(x+\frac{1}{x}\right)^x \left[\frac{x^2-1}{x^2+1} + \log\left(x+\frac{1}{x}\right)\right] + x^{\left(1+\frac{1}{x}\right)}\left[\frac{x^2+1-\log x}{x^2}\right]$

7. Differentiate the functions given

$(\log x)^x + x^{\log x}$

Solution:

Given function is $(\log x)^x + x^{\log x}$

Let $u = (\log x)^x$ and $v = x^{\log x}$, therefore, $y = u + v$

Differentiating with respect to x, we get

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...(i)

Here, $u = (\log x)^x$, taking log on both the sides

$\log u = x \log \log x$, therefore,

$\frac{1}{u}\frac{du}{dx} = x.\frac{d}{dx}\log \log x + \log \log x.\frac{d}{dx}x$

$= x.\frac{1}{\log x}.\frac{1}{x} + \log \log x. 1 = \frac{1}{\log x} + \log \log x$

$\frac{du}{dx} = (\log x)^x \left[\frac{1 + \log x. \log \log x}{\log x}\right]$

$= (\log x)^{x-1}(1 + \log x. \log \log x)$ ...(ii)

and, $v = x^{\log x}$, taking log on both the sides

$\log v = \log x \log x$, therefore,

$\frac{1}{v}\frac{dv}{dx} = \log x.\frac{d}{dx}\log x + logx\,.\frac{d}{dx}\log x$

$= \log x.\frac{1}{x} + \log x.\frac{1}{x}$

$\frac{dv}{dx} = v\left[\frac{2\log x}{x}\right] = x^{\log x}\left[\frac{2\log x}{x}\right] = x^{\log x - 1}(2\log x)$ ...(iii)

Putting the value of $\frac{du}{dx}$ from (ii) and $\frac{dv}{dx}$ from (iii) in equation (i), we get

$\frac{dy}{dx} = (\log x)^{x-1}(1 + \log x. \log \log x) + x^{\log x - 1}(2\log x)$

8. Differentiate the functions given

$(\sin x)^x + \sin^{-1}\sqrt{x}$

Solution:

Given function is $(\sin x)^x + \sin^{-1}\sqrt{x}$

Let $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$, therefore, $y = u + v$

Differentiating with respect to x, we get

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...(i)

Here, $u = (\sin x)^x$, taking log on both the sides

$\log u = x \log \sin x$, therefore,

$\frac{1}{u}\frac{du}{dx} = x.\frac{d}{dx}\log \sin x + \log \sin x.\frac{d}{dx}x$

$= x.\frac{1}{\sin x}.\cos x + \log \sin x. 1 = x \cot x + \log \sin x$

$\frac{du}{dx} = (\sin x)^x (x \cot x + \log \sin x)$ ...(ii)

and, $v = \sin^{-1} \sqrt{x}$, therefore,

$\frac{1}{v}\frac{dv}{dx} = \log x . \frac{d}{dx} \log x + \log x . \frac{d}{dx} \log x$

$= \log x . \frac{1}{x} + \log x . \frac{1}{x}$

$\frac{dv}{dx} = \frac{1}{\sqrt{1-x}} . \frac{d}{dx} \sqrt{x} = \frac{1}{\sqrt{1-x}} . \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x-x^2}}$ ...(iii)

Putting the value of $\frac{du}{dx}$ from (ii) and $\frac{dv}{dx}$ from (iii) in equation (i), we get

$\frac{dy}{dx} = (\sin x)^x (x \cot x + \log \sin x) + \frac{1}{2\sqrt{x-x^2}}$

9. Differentiate the functions given

$x^{\sin x} + (\sin x)^{\cos x}$

Solution:

Given function is $x^{\sin x} + (\sin x)^{\cos x}$

Let $u = x^{\sin x}$ and $v = (\sin x)^{\cos x}$ therefore, $y = u + v$

Differentiating with respect to x, we get

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...(i)

Here, $u = x^{\sin x}$, taking log on both the sides

$\log u = \sin x \log x$, therefore,

$\frac{1}{u}\frac{du}{dx} = \sin x . \frac{d}{dx} \log x + \log x . \frac{d}{dx} \sin x = \sin x . \frac{1}{x} + \log x . \cos x = \frac{\sin x}{x} + \log x \cos x$

$\frac{du}{dx} = x^{\sin x} \left[\frac{\sin x}{x} + \log x \cos x\right] = x^{\sin x - 1}(\sin x + x \log x \cos x)$ ...(ii)

and $v = (\sin x)^{\cos x}$, taking log on both the sides

$\log v = \cos x \log \sin x$, therefore,

$\frac{1}{v}\frac{dv}{dx} = \cos x . \frac{d}{dx} \log \sin x + \log \sin x . \frac{d}{dx} \cos x = \cos x . \frac{1}{\sin x} \cos x + \log \sin x (-\sin x)$

$\frac{dv}{dx} = v[\cos x \cot x - \sin x \log \sin x] = (\sin x)^{\cos x}(\cos x \cot x - \sin x \log \sin x)$ ...(iii)

Putting the value of $\frac{du}{dx}$ from (ii) and $\frac{dv}{dx}$ from (ii) in equation (i), we get

$\frac{dy}{dx} = x^{\sin x - 1}(\sin x + x \log x \cos x) + (\sin x)^{\cos x}(\cos x \cot x - \sin x \log \sin x)$

10. Differentiate the functions given

$x^{x\cos x}+\frac{x^2+1}{x^2-1}$

Solution:

Given function is $x^{x\cos x}+\frac{x^2+1}{x^2-1}$

Let $u=x^{x\cos x}$ and $v=\frac{x^2+1}{x^2-1}$ therefore, $y=u+v$

Differentiating with respect to x, we get

$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$ ...(i)

Here, $u=x^{x\cos x}$, taking log on both the sides

$\log u=x\log x$, therefore,

$\frac{1}{u}\frac{du}{dx}=x\cos x.\frac{d}{dx}\log x+\log x.\frac{d}{dx}x\cos x=x\cos x.\frac{1}{x}+\log x.(-x.\sin x+\cos x)$

$=\cos x-x\sin x\log x+\cos x\log x$

$\frac{du}{dx}=u[\cos x-x\sin x\log x+\cos x\log x]$

$=x^{x\cos x}[\cos x-x\sin x\log x+\cos x\log x]$ ...(ii)

and $v=\frac{x^2+1}{x^2-1}$, taking log on both the sides

$\log v=\log(x^2+1)-\log(x^2-1)$, therefore,

$\frac{1}{v}\frac{dv}{dx}=\frac{1}{x^2+1}\cdot 2x-\frac{1}{x^2-1}\cdot 2x=\frac{2x(x^2-1)-2x(x^2+1)}{(x^2+1)(x^2-1)}=\frac{-4x}{(x^2+1)(x^2-1)}$

$\frac{dv}{dx}=v\left[\frac{-4x}{(x^2+1)(x^2-1)}\right]=\frac{x^2+1}{x^2-1}\left[\frac{-4x}{(x^2+1)(x^2-1)}\right]=-\frac{4x}{(x^2-1)^2}$ ...(iii)

Putting the value of $\frac{du}{dx}$ from (ii) and $\frac{dv}{dx}$ from (iii) in equation (i), we get

$\frac{dy}{dx}=x^{x\cos x}[\cos x-x\sin x\log x+\cos x\log x]-\frac{4x}{(x^2-1)^2}$

11. Differentiate the functions given

$(x\cos x)^x+(x\sin x)^{\frac{1}{x}}$

Solution:

Given function is $(x\cos x)^x+(x\sin x)^{\frac{1}{x}}$

Let $u=(x\cos x)^x$ and $v=(x\sin x)^{\frac{1}{x}}$, therefore, $y=u+v$

Differentiating with respect to x, we get

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...(i)

Here, $u = (x \cos x)^x$, taking log on both the sides

$\log u = x \log(x \cos x)$, therefore,

$\frac{1}{u}\frac{du}{dx} = x.\frac{d}{dx}\log(x \cos x) + \log(x \cos x).\frac{d}{dx}x$

$$= x.\frac{1}{(x \cos x)}(-x \sin x + \cos x) + \log(x \cos x).1 = -x \tan x + 1 + \log(x \cos x)$$

$$\frac{du}{dx} = (x \cos x)^x[1 - x \tan x + \log(x \cos x)]$$

$= (x \cos x)^x[1 - x \tan x + \log(x \cos x)]$ ...(ii)

and, $v = (x \sin x)^{\frac{1}{x}}$, taking log on both the sides

$\log v = \frac{1}{x}\log(x \sin x)$, therefore,

$$\frac{1}{v}\frac{dv}{dx} = \frac{1}{x}.\frac{d}{dx}\log(x \sin x) + \log(x \sin x).\frac{d}{dx}\frac{1}{x}$$

$= \frac{1}{x}.\frac{1}{x \sin x}(x \cos x + \sin x) + \log(x \sin x)\left(-\frac{1}{x^2}\right)$

$$\frac{dv}{dx} = v\left[\frac{x \cot x + 1 - \log(x \sin x)}{x^2}\right]$$

$= (x \sin x)^{\frac{1}{x}}\left[\frac{x \cot x + 1 - \log(x \sin x)}{x^2}\right]$ ...(iii)

Putting the value of $\frac{du}{dx}$ from (ii) and $\frac{dv}{dx}$ from (iii) in equation (i), we get

$\frac{dy}{dx} = (x \cos x)^x[1 - x \tan x + \log(x \cos x)] + (x \sin x)^{\frac{1}{x}}\left[\frac{x \cot x + 1 - \log(x \sin x)}{x^2}\right]$

12. Find $\frac{dy}{dx}$ of the functions given

$x^y + y^x = 1$

Solution:

Given function is $x^y + y^x = 1$

Let $u = x^y$ and $v = y^x$, therefore, $u + v = 1$

Differentiating with respect to x, we get

$\frac{du}{dx} + \frac{dv}{dx} = 0$ ...(i)

Here, $u = x^y$, taking log on both the sides,

$\log u = y \log x$, therefore,

$\frac{1}{u}\frac{du}{dx} = y.\frac{d}{dx}\log x + \log x.\frac{d}{dx}y = y.\frac{1}{x} + \log x.\frac{dy}{dx}$

$\frac{du}{dx} = x^y\left[\frac{y}{x} + \log x.\frac{dy}{dx}\right]$ ...(ii)

and $v = y^x$, taking log on both the sides

$\log v = x \log y$, therefore,

$\frac{1}{v}\frac{dv}{dx} = x.\frac{d}{dx}\log y + \log y.\frac{d}{dx}x = x.\frac{1}{y}\frac{dy}{dx} + \log y.1$

$\frac{dv}{dx} = v\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] = y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right]$ ...(iii)

Putting the value of $\frac{du}{dx}$ from (ii) and $\frac{dv}{dx}$ from (iii) in equation (i), we get

$$x^y\left[\frac{y}{x} + \log x.\frac{dy}{dx}\right] + y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] = 0$$

$$\Rightarrow yx^{y-1} + x^y \log x\frac{dy}{dx} + xy^{x-1}\frac{dy}{dx} + y^x \log y = 0$$

$$\Rightarrow \frac{dy}{dx}(x^y \log x + xy^{x-1}) = -(y^x \log y + yx^{y-1})$$

$\Rightarrow \frac{dy}{dx} = -\frac{y^x \log y + yx^{y-1}}{x^y \log x + xy^{x-1}}$

13. Find $\frac{dy}{dx}$ of the functions given

$y^x = x^y$

Solution:

Given function is $y^x = x^y$

Taking log on both the sides, $x \log y = y \log x$, therefore,

$x.\frac{d}{dx}\log y + \log y.\frac{d}{dx}x = y.\frac{d}{dx}\log x + \log x.\frac{d}{dx}y$

$\Rightarrow x.\frac{1}{y}\frac{dy}{dx} + \log y.1 = y.\frac{1}{x} + \log x.\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}\left(\frac{x}{y} - \log x\right) = \frac{y}{x} - \log y$

$\Rightarrow \frac{dy}{dx}\left(\frac{x - y\log x}{y}\right) = \frac{y - x\log y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{y(y - x\log y)}{x(x - y\log x)}$

14. Find $\frac{dy}{dx}$ of the functions given

$(\cos x)^y = (\cos y)^x$

Solution:

Given function is $(\cos x)^y = (\cos y)^x$

Taking log on both the sides, $y \cos x = x \cos y$, therefore,

$y.\frac{d}{dx}\cos x + \cos x.\frac{d}{dx}y = x.\frac{d}{dx}\cos y + \cos y.\frac{d}{dx}x$

$\Rightarrow y(-\sin x) + \cos x.\frac{dy}{dx} = x.(-\sin y)\frac{dy}{dx} + \cos y.1$

$\Rightarrow \frac{dy}{dx}(\cos x + x \sin y) = \cos y + y \sin x \Rightarrow \frac{dy}{dx} = \frac{\cos y + y \sin x}{\cos x + x \sin y}$

15. Find $\frac{dy}{dx}$ of the functions given

$xy = e^{(x-y)}$

Solution:

Given function is $xy = e^{(x-y)}$

Taking log on both the sides,

$\log x + \log y = (x-y)\log e \Rightarrow \log x + \log y = (x-y)$, therefore,

$\frac{1}{x} + \frac{1}{y}.\frac{dy}{dx} = 1 - \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y} + 1\right) = 1 - \frac{1}{x} \Rightarrow \frac{dy}{dx}\left(\frac{1+y}{y}\right) = \frac{x-1}{x} \Rightarrow \frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$

16. Find the derivative of the function given by $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f'(1)$.

Solution:

Given function is $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$

Taking log on both the sides,

$\log f(x) = \log(1+x) + \log(1+x^2) + \log(1+x^4) + \log(1+x^8)$, therefore,

$\frac{1}{f(x)}.\frac{d}{dx}f(x) = \frac{1}{1+x} + \frac{1}{1+x^2}.\frac{d}{dx}x^2 + \frac{1}{1+x^4}.\frac{d}{dx}x^4 + \frac{1}{1+x^8}.\frac{d}{dx}x^8$

$$\Rightarrow \frac{1}{f(x)}.f'(x) = \frac{1}{1+x} + \frac{1}{1+x^2}.2x + \frac{1}{1+x^4}.4x^3 + \frac{1}{1+x^8}.8x^7$$

$$\Rightarrow f'(x) = f(x)\left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$$

$$\Rightarrow f'(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$$

$$\Rightarrow f'(1) = (1+1)(1+1)(1+1)(1+1)\left[\frac{1}{1+1} + \frac{2}{1+1} + \frac{4}{1+1} + \frac{8}{1+1}\right]$$

$$\Rightarrow f'(1) = (2)(2)(2)(2)\left[\frac{1}{2} + \frac{2}{2} + \frac{4}{2} + \frac{8}{2}\right] = 16\left(\frac{15}{2}\right) = 120$$

17. Differentiate $(x^2 - 5x + 8)(x^3 + 7x + 9)$ in three ways mentioned below:

(i) by using product rule

(ii) by expanding the product to obtain a single polynomial

(iii) by logarithmic differentiation

Do they all give the same answer?

Solution:

Given expression is $(x^2 - 5x + 8)(x^3 + 7x + 9)$

Let $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$

(i) Differentiating using product rule

$$\frac{dy}{dx} = (x^2 - 5x + 8)\frac{d}{dx}(x^3 + 7x + 9) + (x^3 + 7x + 9)\frac{d}{dx}(x^2 - 5x + 8)$$

$$= (x^2 - 5x + 8)(3x^2 + 7) + (x^3 + 7x + 9)(2x - 5)$$

$$= (3x^4 + 7x^2 - 15x^3 - 35x + 24x^2 + 56) + 2x^4 - 5x^3 + 14x^2 - 35x + 18x - 45$$

$$= 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

(ii) Differentiating by expanding the product to obtain a single polynomial

$$y = (x^2 - 5x + 8)(x^3 + 7x + 9)$$

$$= x^5 + 7x^3 + 9x^2 - 5x^4 - 35x^2 - 45x + 8x^3 + 56x + 72$$

$$= x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72$$

$$\frac{dy}{dx} = \frac{d}{dx}x^5 - 5\frac{d}{dx}x^4 + 15\frac{d}{dx}x^3 - 26\frac{d}{dx}x^2 + 11\frac{d}{dx}x + \frac{d}{dx}72$$

$$= 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

(iii) Logarithmic differentiation

$y=(x^2-5x+8)(x^3+7x+9)$

Taking log on both sides, $\log y=\log(x^2-5x+8)+\log(x^3+7x+9)$

$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{(x^2-5x+8)}\cdot\frac{d}{dx}(x^2-5x+8)+\frac{1}{(x^3+7x+9)}\cdot\frac{d}{dx}(x^3+7x+9)$

$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{x^2-5x+8}\cdot(2x-5)+\frac{1}{x^3+7x+9}\cdot(3x^2+7)$

$$\frac{dy}{dx}=y\left[\frac{(2x-5)(x^3+7x+9)+(3x^2+7)(x^2-5x+8)}{(x^2-5x+8)(x^3+7x+9)}\right]$$

$$=y\left[\frac{2x^4+14x^2+18x-5x^3-35x-45+3x^4-15x^3+24x^2+7x^2-35x+56}{(x^2-5x+8)(x^3+7x+9)}\right]$$

$\Rightarrow\frac{dy}{dx}=(x^2-5x+8)(x^3+7x+9)\left[\frac{5x^5-20x^3+45x^2-52x+11}{(x^2-5x+8)(x^3+7x+9)}\right]$

$\Rightarrow\frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$

Hence, all the three answers are same.

18. If u, v and w are functions of x, then show that

$\frac{d}{dx}(u,v,w)=\frac{du}{dx}v.w+u.\frac{dv}{dx}.w+u.v\frac{dw}{dx}$ in two ways – first by repeated application of product rule, second by logarithmic differentiation.

Solution:

Given that u, v and w are functions of x

Let $y=u.v.w=u.(v.w)$

Differentiation by repeated application of product rule

$\frac{dy}{dx}=u.\frac{d}{dx}(v.w)+(v.w).\frac{d}{dx}u$

$=u\left[v\frac{d}{dx}w+w\frac{d}{dx}v\right]+v.w.\frac{du}{dx}$

$\Rightarrow\frac{dy}{dx}=u.v.\frac{dw}{dx}+u.w.\frac{dv}{dx}+v.w.\frac{du}{dx}$

Differentiation using logarithmic

Let $y=u.v.w$

Taking log on both the sides, $\log y=\log u+\log v+\log w$

$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{u}\cdot\frac{du}{dx}+\frac{1}{v}\cdot\frac{dv}{dx}+\frac{1}{w}\cdot\frac{dw}{dx}$

$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{u}.\frac{du}{dx} + \frac{1}{v}.\frac{dv}{dx} + \frac{1}{w}.\frac{dw}{dx}\right]$

$\Rightarrow \frac{dy}{dx} = u.v.w\left[\frac{1}{u}.\frac{du}{dx} + \frac{1}{v}.\frac{dv}{dx} + \frac{1}{w}.\frac{dw}{dx}\right]$

$\Rightarrow \frac{dy}{dx} = \frac{u.v.w}{u}.\frac{du}{dx} + \frac{u.v.w}{v}.\frac{dv}{dx} + \frac{u.v.w}{w}.\frac{dw}{dx}$

$\Rightarrow \frac{dy}{dx} = v.w.\frac{du}{dx} + u.w.\frac{dv}{dx} + u.v.\frac{dw}{dx}$

Exercise 5.6

1. If x and y are connected parametrically by the equations given in this question, without eliminating the parameter. Find $\frac{dy}{dx}$.

$x = 2at^2, y = at^4$

Solution:

Given that x and y are connected parametrically and here, $x = 2at^2, y = at^4$

Therefore, $\frac{dx}{dt} = 2a(2t)$ and $\frac{dy}{dt} = a(4t^3)$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4at^3}{4at} = t^2$

2. If x and y are connected parametrically by the equations given in this question, without eliminating the parameter. Find $\frac{dy}{dx}$.

$x = a\cos\theta, y = b\cos\theta$

Solution:

Given that x and y are connected parametrically and here, $x = a\cos\theta, y = b\cos\theta$

Therefore, $\frac{dx}{d\theta} = a(-\sin\theta)$ and $\frac{dy}{d\theta} = b(-\sin\theta)$

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$

3. If x and y are connected parametrically by the equations given in this question, without eliminating the parameter. Find $\frac{dy}{dx}$.

$x = \sin t, y = \cos 2t$

Solution:

Given that x and y are connected parametrically and here, $x = \sin t, y = \cos 2t$

Therefore, $\frac{dx}{dt} = \cos t$ and $\frac{dy}{dt} = -\sin 2t.2$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2\sin 2t}{\cot t} = -\frac{2(2\sin t\cos t)}{\cos t} = -4\sin t$$

4. If x and y are connected parametrically by the equations given in this question, without eliminating the parameter. Find $\frac{dy}{dx}$.

$x = 4t, y = \frac{4}{t}$

Solution:

Given that x and y are connected parametrically and here, $x = 4t, y = \frac{4}{t}$

Therefore, $\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = -\frac{4}{t^2}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\frac{4}{t^2}}{4} = -\frac{1}{t^2}$$

5. If x and y are connected parametrically by the equations given in this question, without eliminating the parameter. Find $\frac{dy}{dx}$.

$x = \cos\theta - \cos 2\theta, y = \sin\theta - \sin 2\theta$

Solution:

Given that x and y are connected parametrically and here, $x = \cos\theta - \cos 2\theta, y = \sin\theta - \sin 2\theta$

Therefore, $\frac{dx}{d\theta} = -\sin\theta + 2\sin 2\theta$ and $\frac{dy}{d\theta} = \cos\theta - 2\cos 2\theta$

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos\theta - 2\cos 2\theta}{-\sin\theta + 2\sin 2\theta}$

6. If x and y are connected parametrically by the equations given in this question, without eliminating the parameter. Find $\frac{dy}{dx}$.

$x = a(\theta - \sin\theta), y = a(1 + \cos\theta)$

Solution:

Given that x and y are connected parametrically and here, $x = a(\theta - \sin\theta), y = a(1 + \cos\theta)$

Therefore, $\frac{dx}{d\theta} = a(1 - \cos\theta)$ and $\frac{dy}{d\theta} = a(0 - \sin\theta)$

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{(-a\sin\theta)}{a(1-\cos\theta)} = -\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = -\cot\frac{\theta}{2}$

7. If x and y are connected parametrically by the equations given in this question, without eliminating the parameter. Find $\frac{dy}{dx}$.

$x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

Solution:

Given that x and y are connected parametrically and here, $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

Therefore, $\frac{dx}{dt} = \frac{\sin^3 t\frac{d}{dt}\sqrt{\cos 2t} - \sqrt{\cos 2t}\frac{d}{dt}\sin^3 t}{\left(\sqrt{\cos 2t}\right)^2}$

$= \frac{\sin^3 t.\frac{1}{2\sqrt{\cos 2t}}.(-\sin 2t).2 - \sqrt{\cos 2t}\,.3\sin^2 t\cos t}{\cos 2t}$

$= \frac{-\sin^3 t.\sin 2t - 3\cos 2t.\sin^2 t\cos t}{\cos 2t\sqrt{\cos 2t}}$

and $\frac{dy}{dt} = \frac{\cos^3 t\frac{d}{dt}\sqrt{\cos 2t} - \sqrt{\cos 2t}\frac{d}{dt}\cos^3 t}{\left(\sqrt{\cos 2t}\right)^2}$

$$= \frac{\cos^3 t.\frac{1}{2\sqrt{\cos 2t}}(-\sin 2t).2 - \sqrt{\cos 2t}.3\cos^2 t(-\sin t)}{\cos 2t}$$

$$= \frac{-\cos^3 t.\sin 2t + 3\cos 2t.\cos^2 t\sin t}{\cos 2t\sqrt{\cos 2t}}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\cos^3 t.\sin 2t + 3\cos 2t.\cos^2 t\sin t}{-\sin^3 t.\sin 2t - 3\cos 2t.\sin^2 t\cos t}$$

$$= \frac{-\cos^3 t.(2\sin t\cos t) + 3\cos 2t.\cos^2 t\sin t}{-\sin^3 t.(2\sin t\cos t) - 3\cos 2t.\sin^2 t\cos t} = \frac{\cos^2 t\sin t(-2\cos^2 t + 3\cos 2t)}{\sin^2 t\cos t(-2\sin^2 t - 3\cos 2t)}$$

$$= \frac{\cos t[-2\cos^2 t + 3(2\cos^2 - 1)]}{\sin t[-2\sin^2 t - 3(1 - 2\sin^2 t)]} = \frac{\cos t[-2\cos^2 t + 6\cos^2 t - 3]}{\sin t[-2\sin^2 t - 3 + 6\sin^2 t]}$$

$$= \frac{\cos t[4\cos^2 t - 3]}{\sin t[-3 + 4\sin^2 t]} = -\frac{4\cos^3 t - 3\cos t}{3\sin t - 4\sin^3 t} = -\frac{\cos 3t}{\sin 3t} = -\cot 3t$$

8. If x and y are connected parametrically by the equations given in this question, without eliminating the parameter. Find $\frac{dy}{dx}$.

$x = a\left(\cos t + \log\tan\frac{t}{2}\right) y = a\sin t$

Solution:

Given that x and y are connected parametrically and here, $x = a\left(\cos t + \log\tan\frac{t}{2}\right) y = a\sin t$

Therefore, $\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan\frac{t}{2}}.\sec^2\frac{t}{2}.\frac{1}{2}\right) = a\left(-\sin t + \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}.\frac{1}{\cos^2\frac{t}{2}}.\frac{1}{2}\right)$

$$= a\left(-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right) = a\left(-\sin t + \frac{1}{\sin t}\right) = a\left(\frac{-\sin^2 t + 1}{\sin t}\right) = a\left(\frac{\cos^2 t}{\sin t}\right)$$

$$\frac{dy}{dt} = a\cos t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{a\left(\frac{\cos^2 t}{\sin t}\right)} = \frac{\sin t}{\cos t} = \tan t$$

9. If x and y are connected parametrically by the equations given in this question, without eliminating the parameter. Find $\frac{dy}{dx}$.

$x = a\sec\theta, y = b\tan\theta$

Solution:

Given that x and y are connected parametrically and here, $x = a\sec\theta, y = b\tan\theta$

Therefore, $\frac{dx}{d\theta} = a \sec\theta \tan\theta$

and $\frac{dy}{d\theta} = b \sec^2\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b\sec\theta}{a\tan\theta} = \frac{b\left(\frac{1}{\cos\theta}\right)}{a\left(\frac{\sin\theta}{\cos\theta}\right)} = \frac{b}{a}\operatorname{cosec}\theta$$

10. If x and y are connected parametrically by the equations given in this question, without eliminating the parameter. Find $\frac{dy}{dx}$.

$x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$

Solution:

Given that x and y are connected parametrically and here, $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$

Therefore, $\frac{dx}{d\theta} = a[-\sin\theta + (\theta\cos\theta + \sin\theta)] = a\theta\cos\theta$

and $\frac{dy}{d\theta} = a[\cos\theta - (-\theta\sin\theta + \cos\theta)] = a\theta\sin\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

11. If $x = \sqrt{a^{\sin^{-1}t}}, y = \sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$

Solution:

Given that x and y are connected parametrically and here, $x = \sqrt{a^{\sin^{-1}t}}, y = \sqrt{a^{\cos^{-1}t}}$

Therefore,

$$\frac{dx}{dt} = \frac{1}{2\sqrt{a^{\sin^{-1}t}}}.\frac{d}{dx}a^{\sin^{-1}t} = \frac{1}{2\sqrt{a^{\sin^{-1}t}}}.a^{\sin^{-1}t}.\log a\frac{1}{\sqrt{1-t^2}}$$

$$= \frac{1}{2x}.x^2.\log a\frac{1}{\sqrt{1-t^2}} = \frac{x\log a}{\sqrt{1-t^2}}$$

and

$$\frac{dy}{dt} = \frac{1}{2\sqrt{a^{\cos^{-1}t}}}.\frac{d}{dx}a^{\cos^{-1}t} = \frac{1}{2\sqrt{a^{\cos^{-1}t}}}.a^{\cos^{-1}t}.\log a\frac{-1}{\sqrt{1-t^2}}$$

$$= \frac{1}{2y}.y^2.\log a\frac{1}{\sqrt{1-t^2}} = -\frac{y\log a}{\sqrt{1-t^2}}$$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\frac{y \log a}{\sqrt{1-t^2}}}{\frac{x \log a}{\sqrt{1-t^2}}} = -\frac{y}{x}$

Exercise 5.7

1. Find the second order derivatives of the function given

$x^2 + 3x + 2$

Solution:

Given function is $x^2 + 3x + 2$

Let $y = x^2 + 3x + 2$, therefore,

$\frac{dy}{dx} = \frac{d}{dx}(x^2 + 3x + 2) = 2x + 3$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(2x + 3) = 2$

Second order derivative of $x^2 + 3x + 2 = 2$

2. Find the second order derivatives of the function given

x^{20}

Solution:

Given function is x^{20}

Let $y = x^{20}$, therefore,

$\frac{dy}{dx} = \frac{d}{dx}(x^{20}) = 20x^{19}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(20x^{19}) = 380x^{18}$

3. Find the second order derivatives of the function given

$x.\cos x$

Solution:

Given function is $x.\cos x$

Let $y = x.\cos x$, therefore,

$\frac{dy}{dx} = \frac{d}{dx}(x.\cos x) = x.\frac{d}{dx}\cos x + \cos x.\frac{d}{dx}x = -x\sin x + \cos x$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(-x\sin x + \cos x) = -\left(x\frac{d}{dx}\sin x + \sin x\frac{d}{dx}x\right) - \sin x$

$= -x\cos x - \sin x - \sin x = -(x\cos x + 2\sin x)$

4. Find the second order derivatives of the function given

$\log x$

Solution:

Given function is $\log x$

Let $y = \log x$, therefore,

$\frac{dy}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

5. Find the second order derivatives of the function given

$x^3\log x$

Solution:

Given function is $x^3\log x$

Let $y = x^3\log x$, therefore,

$\frac{dy}{dx} = \frac{d}{dx}(x^3\log x) = x^3.\frac{d}{dx}\log x + \log x.\frac{d}{dx}x^3 = x^3.\frac{1}{x} + \log x.3x^2 = x^2 + 3x^2\log x$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(x^2 + 3x^2\log x) = 2x + 3\left(x^2\frac{d}{dx}\log x + \log x\frac{d}{dx}x^2\right)$

$= 2x + 3\left(x^2.\frac{1}{x} + \log x.2x\right) = 2x + 3x + 6x\log x = 5x + 6x\log x = x(5 + 6\log x)$

6. Find the second order derivatives of the function given

$e^x \sin 5x$

Solution:

Given function is $e^x \sin 5x$

Let $y = e^x \sin 5x$, therefore,

$\frac{dy}{dx} = \frac{d}{dx}(e^x \sin 5x) = e^x.\frac{d}{dx}\sin 5x + \sin 5x.\frac{d}{dx}e^x = e^x.\cos 5x.5 + \sin 5x.e^x$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(5e^x \cos 5x + e^x \sin 5x)$

$= 5\left(e^x.\frac{d}{dx}\cos 5x + \cos 5x.\frac{d}{dx}e^x\right) + \left(e^x.\frac{d}{dx}\sin 5x + \sin 5x.\frac{d}{dx}e^x\right)$

$= 5[e^x.(-\sin 5x).5 + \cos 5x.e^x] + [e^x.\cos 5x.5 + \sin 5x.e^x]$

$= e^x(-25\sin 5x + 5\cos 5x + 5\cos 5x + \sin 5x) = e^x(10\cos 5x - 24\sin 5x)$

7. Find the second order derivatives of the function given

$e^{6x}\cos 3x$

Solution:

Given function is $e^{6x}\cos 3x$

Let $y = e^{6x}\cos 3x$, therefore,

$\frac{dy}{dx} = \frac{d}{dx}(e^{6x}\cos 3x) = e^{6x}.\frac{d}{dx}\cos 3x + \cos 3x.\frac{d}{dx}e^{6x}$

$= e^{6x}.(-\sin 3x).3 + \cos 3x.e^{6x}.6 = 3e^{6x}(-\sin 3x + 2\cos 3x)$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}[3e^{6x}(-\sin 3x + 2\cos 3x)]$

$= 3e^{6x}.\frac{d}{dx}(-\sin 3x + 2\cos 3x) + (-\sin 3x + 2\cos 3x).\frac{d}{dx}3e^{6x}$

$= 3e^{6x}.(-3\cos 3x - 6\sin 3x) + (-\sin 3x + 2\cos 3x).18e^{6x}$

$= e^{6x}(-9\cos 3x - 18\sin 3x - 18\sin 3x + 36\cos 3x)$

$= e^{6x}(27\cos 3x - 36\sin 3x)$

$= 9e^{6x}(3\cos 3x - 4\sin 3x)$

8. Find the second order derivatives of the function given

$\tan^{-1} x$

Solution:

Given function is $\tan^{-1} x$

Let $y = \tan^{-1} x$, therefore,

$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{(1+x^2)\frac{d}{dx}1 - 1.\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$= \frac{0-2x}{(1+x^2)^2} = -\frac{2x}{(1+x^2)^2}$

9. Find the second order derivatives of the function given

$\log(\log x)$

Solution:

Given function is $\log(\log x)$

Let $y = \log(\log x)$, therefore,

$\frac{dy}{dx} = \frac{d}{dx}(\log(\log x)) = \frac{1}{\log x}.\frac{1}{x} = \frac{1}{x\log x}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x\log x}\right) = \frac{(x\log x)\frac{d}{dx}1 - 1.\frac{d}{dx}(x\log x)}{(x\log x)^2}$$

$= \frac{0-\left(x.\frac{1}{x}+\log x\right)}{(x\log x)^2} = -\frac{1+\log x}{(x\log x)^2}$

10. Find the second order derivatives of the function given

$\sin(\log x)$

Solution:

Given function is $\sin(\log x)$

Let $y = \sin(\log x)$, therefore,

$\frac{dy}{dx} = \frac{d}{dx}(\sin(\log x)) = \cos(\log x).\frac{1}{x} = \frac{\cos(\log x)}{x}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{\cos(\log x)}{x}\right] = \frac{x\frac{d}{dx}\cos(\log x) - \cos(\log x).\frac{d}{dx}x}{(x)^2}$$

$$= \frac{x\{-\sin(\log x).\frac{1}{x}\} - \cos(\log x).1}{(x)^2} = \frac{-\sin(\log x) - \cos(\log x)}{(x)^2}$$

11. If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$

Solution:

Given that $y = 5\cos x - 3\sin x$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(5\cos x - 3\sin x) = -5\sin x - 3\cos x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(-5\sin x - 3\cos x) = -5\cos x + 3\sin x = -(5\cos x - 3\sin x) = -y$$

$$\Rightarrow \frac{d^2y}{dx^2} + y = 0$$

12. If $y = \cos^{-1} x$, find $\frac{d^2y}{dx^2}$ in terms of y alone.

Solution:

Given that $y = \cos^{-1} x \Rightarrow \cos y = x$, therefore,

$$-\sin y\frac{dy}{dx} = 1 \quad \Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y} = -\operatorname{cosec} y$$

$$\Rightarrow \frac{d^2y}{dx^2} = -(\operatorname{cosec} y \cot y).\frac{dy}{dx} = (\operatorname{cosec} y \cot y).(-\operatorname{cosec} y) = -\operatorname{cosec}^2 y \cot y$$

13. If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2y_2 + xy_1 + y = 0$

Solution:

Given that $y = 3\cos(\log x) + 4\sin(\log x)$, therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(3\cos(\log x) + 4\sin(\log x)) = -3\sin(\log x).\frac{1}{x} + 4\cos(\log x).\frac{1}{x}$$

$\Rightarrow x\dfrac{dy}{dx} = -3\sin(\log x) + 4\cos(\log x)$

$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx}.\frac{d}{dx}x = \frac{d}{dx}[-3\sin(\log x) + 4\cos(\log x)]$

$= -3\cos(\log x).\frac{1}{x} - 4\sin(\log x).\frac{1}{x} = -\frac{1}{x}[3\cos(\log x) + 4\sin(\log x)] = -\frac{1}{x}.y$

$\Rightarrow x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = -\dfrac{1}{x}y \quad \Rightarrow x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = -y \quad \Rightarrow x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$

$\Rightarrow x^2y_2 + xy_1 + y = 0$

14. If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

Solution:

Given that $y = Ae^{mx} + Be^{nx}$, therefore,

$\dfrac{dy}{dx} = \dfrac{d}{dx}(Ae^{mx} + Be^{nx}) = mAe^{mx} + nBe^{nx}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(mAe^{mx} + nBe^{nx}) = m^2Ae^{mx} + n^2Be^{nx}$

Putting the value of $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ in $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$, we get

$\text{LHS} = (m^2Ae^{mx} + n^2Be^{nx}) - (m+n)(mAe^{mx} + nBe^{nx}) + mny$

$= m^2Ae^{mx} + n^2Be^{nx} - (m^2Ae^{mx} + mnBe^{nx} + mnAe^{mx} + n^2Be^{nx}) + mny$

$= -(mnAe^{mx} + mnBe^{nx}) + mny$

$= -mn(Ae^{mx} + Be^{nx}) + mny$

$= -mny + mny = 0 = \text{RHS}$

15. If $y = 500e^{7x} + 600e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$.

Solution:

Given that $y = 500e^{7x} + 600e^{-7x}$, therefore,

$\frac{dy}{dx} = \frac{d}{dx}(500e^{7x} + 600e^{-7x}) = 500e^{7x}.7 + 600e^{-7x}.(-7) = 7(500e^{7x} - 600e^{-7x})$

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}7(500e^{7x} - 600e^{-7x}) = 7[500e^{7x}.7 + 600e^{-7x}.(-7)]$

$= 49(500e^{7x} - 600e^{-7x}) = 49y$

$\Rightarrow \frac{d^2y}{dx^2} = 49y$

16. If $e^y(x+1) = 1$, show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

Solution:

Given that $e^y(x+1) = 1$, therefore,

$$e^y\frac{dy}{dx}(x+1) + (x+1)\frac{d}{dx}e^y = \frac{d}{dx}1$$

$$\Rightarrow e^y + (x+1)e^y\frac{dy}{dx} = 0$$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{x+1}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{1}{x+1}\right) = -\left[\frac{(x+1).\frac{d}{dx}1 - 1.\frac{d}{dx}(x+1)}{(x+1)^2}\right] = -\left[\frac{0-1}{(x+1)^2}\right] = \frac{1}{(x+1)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left(-\frac{1}{x+1}\right)^2$$

$\Rightarrow \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

17. If $y = (\tan^{-1}x)^2$, show that $(x^2+1)^2y_2 + 2x(x^2+1)y_1 = 2$.

Solution:

Given that $y = (\tan^{-1}x)^2$, therefore,

$\frac{dy}{dx} = \frac{d}{dx}[(\tan^{-1}x)^2] = 2\tan^{-1}x.\frac{1}{1+x^2} = \frac{2\tan^{-1}x}{1+x^2}$

$$\Rightarrow (1+x^2)\frac{dy}{dx} = 2\tan^{-1}x$$

$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + \frac{dy}{dx}.\frac{d}{dx}(1+x^2) = \frac{d}{dx}(2\tan^{-1}x)$

$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + \frac{dy}{dx}.2x = \frac{2}{1+x^2}$

$$\Rightarrow (1+x^2)^2\frac{d^2y}{dx^2}+2x(1+x^2)\frac{dy}{dx}=2$$

$$\Rightarrow (x^2+1)^2y_2+2x(x^2+1)y_1=2$$

Exercise 5.8

1. Verify Rolle's Theorem for the function $f(x)=x^2+2x-8, x\in[-4,2]$.

Solution:

Given function is $f(x)=x^2+2x-8, x\in[-4,2]$

(i) Function f is a polynomial function, so it is continuous in close interval $[-4,2]$.

(ii) $f'(x)=2x+2$

Hence, the function f is differentiable in open interval $(-4,2)$.

(iii) $f(-4)=(-4)^2+2(-4)-8=16-8-8=0$

and $f(2)=(2)^2+2(2)-8=4+4-8=0$

$\Rightarrow f(-4)=f(2)$

Here, all the three conditions of Rolle's Theorem is satisfied. Therefore, there must be a number $c\in(-4,2)$ such that $f'(c)=0$.

$\Rightarrow f'(c)=2c+2=0$

$\Rightarrow c=-1\in(-4,2)$

Hence, the Rolle's Theorem is verified for the function $f(x)=x^2+2x-8, x\in[-4,2]$.

2. Examine if Rolle's Theorem is applicable to any of the following functions. Can you say something about the converse of Rolle's Theorem from these example?

(i) $f(x)=[x]$ for $x\in[5,9]$

(ii) $f(x)=[x]$ for $x\in[-2,2]$

(iii) $f(x)=x^2-1$ for $x\in[1,2]$

Solution:

Rolle's Theorem is applicable to function $f:[a,b]\to R$ the following three conditions of Rolle's Theorem is satisfied.

(i) Function f is continuous in closed interval $[a,b]$.

(ii) Function f is differentiable in open interval (a, b).

(iii) $f(a) = f(b)$

(i) $f(x) = [x]$ for $x \in [5, 9]$

The greatest integer function f is neither continuous in close interval $[5, 9]$ nor differentiable in open interval $(5, 9)$. Also $f(5) \neq f(9)$.

Hence, the Rolle's Theorem is not applicable to $f(x) = [x]$ for $x \in [5, 9]$.

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

The greatest integer function f is neither continuous in close interval $[-2, 2]$ nor differentiable in open interval $(2, 2)$. Also $f(-2) \neq f(2)$.

Hence, the Rolle's Theorem is not applicable to $f(x) = [x]$ for $x \in [-2, 2]$.

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

The function f is a polynomial function, so it is continuous in closed interval $[1, 2]$.

$f'(x) = 2x$, hence, the function f is differentiable in open interval $(1, 2)$.

$f(1) = (1)^2 - 1 = 0$ and

$f(2) = (2)^2 - 1 = 3$,

$\Rightarrow f(1) \neq f(2)$

Hence, Rolle's Theorem is not applicable to the function $f(x) = x^2 - 1$ for $x \in [1, 2]$.

3. If $f: [-5, 5] \to \boldsymbol{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$

Solution:

$f: [-5, 5] \to \boldsymbol{R}$ is a differentiable function, hence

(i) The function f is continuous in closed interval $[-5, 5]$.

(ii) The function f is continuous in open interval $(-5, 5)$.

According to Mean Value Theorem, there exists a value $c \in (-5, 5)$, such that

$$f'(c) = \frac{f(5) - f(-5)}{5 - (-5)}$$

But it is given that $f'(x)$ does not vanish anywhere, hence

$$f'(c) = \frac{f(5) - f(-5)}{5 - (-5)} \neq 0$$

$\Rightarrow f(5) - f(-5) \neq 0$

$\Rightarrow f(5) \neq f(-5)$

4. Verify Mean Value Theorem, if $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$, where $a = 1$ and $b = 4$.

Solution:

Given function is $f(x) = x^2 - 4x - 3, x \in [1, 4]$

(i) Function f is a polynomial function, hence it is continuous in closed interval $[1, 4]$.

(ii) $f'(x) = 2x - 4$

Hence, the function f is differentiable in open interval $(1, 4)$.

According to Mean Value Theorem, there exists a value $c \in (1, 4)$, such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}$$

$$\Rightarrow 2c - 4 = \frac{[(4)^2 - 4(4) - 3] - [(1)^2 - 4(1) - 3]}{3}$$

$$\Rightarrow 2c - 4 = \frac{-3 - (-6)}{3} = \frac{3}{3} = 1$$

$$\Rightarrow 2c = 5 \quad \Rightarrow c = \frac{5}{2} \in (1, 4)$$

Hence, for the function $f(x) = x^2 - 4x - 3, x \in [1, 4]$, the Mean Value Theorem is verified.

5. Verify Mean Value Theorem, if $f(x) = x^3 - 5x^2 - 3x$ in the interval $[a, b]$, where $a = 1$ and $b = 3$. Find all $c \in (1, 3)$ for which $f'(c) = 0$.

Solution:

Given function is $f(x) = x^3 - 5x^2 - 3x, x \in [1, 3]$

(i) Function f is a polynomial function, hence it is continuous in closed interval $[1, 3]$.

(ii) $f'(x) = 3x^2 - 10x - 3$

Hence, the function f is differentiable in open interval $(1, 3)$.

According to Mean Value Theorem, there exists a value $c \in (1, 3)$, such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$\Rightarrow 3c^2 - 10c - 3 = \frac{[(3)^3 - 5(3)^2 - 3(3)] - [(1)^3 - 5(1)^2 - 3(1)]}{2}$$

$$\Rightarrow 3c^2 - 10c - 3 = \frac{(27 - 54) - (1 - 8)}{2} = \frac{-27 + 7}{2} = -10$$

$\Rightarrow 3c^2 - 10c + 7 = 0$

$\Rightarrow 3c^2 - 3c - 7c + 7 = 0$

$\Rightarrow 3c(c-1) - 7(c-1) = 0$

$\Rightarrow (c-1)(3c-7) = 0$

$\Rightarrow c - 1 = 0$ or $3c - 7 = 0$

$\Rightarrow c = 1$ or $c = \frac{7}{3}$

$\Rightarrow c = \frac{7}{3} \in (1,3)$

Hence, for the function $f(x) = x^3 - 5x^2 - 3x, x \in [1,3]$, the Mean Value Theorem is verified. For the value of $c = \frac{7}{3}$ the function $f'(c) = 0$.

6. Examine the applicability of Mean Value Theorem for all three functions

Solution:

Mean Value Theorem is applicable to function $f: [a, b] \to R$ the following two conditions of Mean Value Theorem is satisfied.

(i) Function f is continuous in closed interval $[a, b]$.

(ii) Function f is differentiable in open interval (a, b).

(i) $f(x) = [x]$ for $x \in [5, 9]$

The greatest integer function f is neither continuous in close interval $[5, 9]$ nor differentiable in open interval $(5, 9)$.

Hence, the Mean Value Theorem is not applicable to $f(x) = [x]$ for $x \in [5, 9]$.

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

The greatest integer function f is neither continuous in close interval $[-2, 2]$ nor differentiable in open interval $(2, 2)$.

Hence, the Mean Value Theorem is not applicable to $f(x) = [x]$ for $x \in [-2, 2]$.

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

The function f is polynomial function, so it is continuous in closed interval $[1, 2]$.

$f'(x) = 2x$, hence, the function f is differentiable in open interval $(1, 2)$.

Hence, Mean Value Theorem is not applicable to the function $f(x) = x^2 - 1$ for $x \in [1, 2]$.

Hence, the Mean Value Theorem is applicable to $f(x) = x^2 - 1$ for $x \in [1, 2]$.

Miscellaneous

1. Differentiate w.r.t. x the function

$(3x^2 - 9x + 5)^9$

Solution:

Given function is $(3x^2 - 9x + 5)^9$

Let $y = (3x^2 - 9x + 5)^9$, therefore,

$\frac{dy}{dx} = 9(3x^2 - 9x + 5)^8 . \frac{d}{dx}(3x^2 - 9x + 5) = 9(3x^2 - 9x + 5)^8 . (6x - 9)$

$= 27(3x^2 - 9x + 5)^8 . (2x - 3)$

2. Differentiate w.r.t. x the function

$\sin^3 x + \cos^6 x$

Solution:

Given function is $\sin^3 x + \cos^6 x$

Let $y = \sin^3 x + \cos^6 x$, therefore,

$\frac{dy}{dx} = 3\sin^2 x . \frac{d}{dx}\sin x + 6\cos^5 x . \frac{d}{dx}\cos x = 3\sin^2 x . \cos x + 6\cos^5 x . (-\sin x)$

$= 3\sin x \cos x (\sin x - 2\cos^4 x)$

3. Differentiate w.r.t. x the function

$(5x)^{3\cos 2x}$

Solution:

Given function is $(5x)^{3\cos 2x}$

Let $y = (5x)^{3\cos 2x}$, taking log on both sides

$\log y = \log(5x)^{3\cos 2x} = 3\cos 2x . \log 5x$

Therefore,

$$\frac{1}{y}\frac{dy}{dx} = 3\cos 2x.\frac{d}{dx}\log 5x + \log 5x.\frac{d}{dx}3\cos 2x$$

$$\Rightarrow \frac{dy}{dx} = y\left[3\cos 2x.\frac{1}{5x}.5 + \log 5x.3(-\sin 2x).2\right]$$

$$\Rightarrow \frac{dy}{dx} = 3(5x)^{3\cos 2x}\left[\frac{\cos 2x - 2\sin 2x\log 5x}{x}\right]$$

4. Differentiate w.r.t. x the function

$\sin^{-1}(x\sqrt{x}), 0 \le x \le 1$

Solution:

Given function is $\sin^{-1}(x\sqrt{x}), 0 \le x \le 1$

Let $y = \sin^{-1}(x\sqrt{x})$, therefore,

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x\sqrt{x})^2}}.\frac{d}{dx}(x\sqrt{x}) = \frac{1}{\sqrt{1-x^3}}.\left[x\frac{d}{dx}\sqrt{x} + \sqrt{x}.\frac{d}{dx}x\right]$$

$$= \frac{1}{\sqrt{1-x^3}}\cdot\left[x\frac{1}{2\sqrt{x}} + \sqrt{x}\cdot 1\right] = \frac{1}{\sqrt{1-x^3}}\cdot\left[\frac{x+2x}{2\sqrt{x}}\right] = \frac{3x}{2\sqrt{x}\sqrt{1-x^3}} = \frac{3}{2}\sqrt{\frac{x}{1-x^3}}$$

5. Differentiate w.r.t. x the function

$\frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}, -2 < x < 2.$

Solution:

Given function is $\frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}, -2 < x < 2$

Let $y = \frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}$, therefore,

$$\frac{dy}{dx} = \frac{\cos^{-1}\frac{x}{2}.\frac{d}{dx}\sqrt{2x+7} - \sqrt{2x+7}\frac{d}{dx}\cos^{-1}\frac{x}{2}}{\left(\sqrt{2x+7}\right)^2}$$

$$= \frac{\left[\cos^{-1}\frac{x}{2}.\frac{1}{2\sqrt{2x+7}}.2\right] - \sqrt{2x+7}\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}.\frac{1}{2}}{2x+7}$$

$$= \frac{\cos^{-1}\frac{x}{2}.\frac{1}{\sqrt{2x+7}}+\sqrt{2x+7}\frac{1}{\sqrt{4-(x)^2}}}{2x+7} = \frac{\cos^{-1}\frac{x}{2}.\sqrt{4-x^2}+2x+7}{(2x+7)\sqrt{2x+7}\sqrt{4-x^2}}$$

6. Differentiate w.r.t. x the function

$\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right], 0 < x < \frac{\pi}{2}$

Solution:

Given function is $\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right], 0 < x < \frac{\pi}{2}$

Let $y = \cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right]$, therefore,

$$y = \cot^{-1}\left[\frac{\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}+\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}{\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}-\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}\right]$$

$$= \cot^{-1}\left[\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right]$$

$$= \cot^{-1}\left[\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}\right] = \cot^{-1}\left[\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right] = \cot^{-1}\left[\cot\frac{x}{2}\right] = \frac{x}{2}$$

Therefore, $\frac{dy}{dx} = \frac{x}{2}$

7. Differentiate w.r.t. x the function

$(\log x)^{\log x}, x > 1$

Solution:

Given function is $(\log x)^{\log x}, x > 1$

Let $y = (\log x)^{\log x}$, taking log on both sides

$\log y = \log(\log x)^{\log x} = \log x.\log(\log x)$

Therefore,

$$\frac{1}{y}\frac{dy}{dx} = \log x.\frac{d}{dx}\log(\log x) + \log(\log x).\frac{d}{dx}\log x$$

$$\Rightarrow \frac{dy}{dx} = y\left[\log x.\frac{1}{\log x}.\frac{1}{x} + \log(\log x).\frac{1}{x}\right]$$

$$\Rightarrow \frac{dy}{dx} = (\log x)^{\log x}\left[\frac{1+\log(\log x)}{x}\right]$$

8. Differentiate w.r.t. x the function

$\cos(a\cos x + b\sin x)$, for some constant a and b.

Solution:

Given function is $\cos(a\cos x + b\sin x)$

Let $y = \cos(a\cos x + b\sin x)$, therefore,

$$\frac{dy}{dx} = -\sin(a\cos x + b\sin x).\frac{d}{dx}(a\cos x + b\sin x)$$

$$= -\sin(a\cos x + b\sin x)(-a\sin x + b\cos x)$$

$$= \sin(a\cos x + b\sin x)\,(a\sin x - b\cos x)$$

9. Differentiate w.r.t. x the function

$(\sin x - \cos x)^{(\sin x - \cos x)}, \frac{\pi}{4} < x < \frac{3\pi}{4}$

Solution:

Given function is $(\sin x - \cos x)^{(\sin x - \cos x)}, \frac{\pi}{4} < x < \frac{3\pi}{4}$

Let $y = (\sin x - \cos x)^{(\sin x - \cos x)}$, taking log on both sides

$$\log y = \log(\sin x - \cos x)^{(\sin x - \cos x)} = (\sin x - \cos x).\log(\sin x - \cos x)$$

Therefore,

$$\frac{1}{y}\frac{dy}{dx} = (\sin x - \cos x).\frac{d}{dx}\log(\sin x - \cos x) + \log(\sin x - \cos x).\frac{d}{dx}(\sin x - \cos x)$$

$$\Rightarrow \frac{dy}{dx} = y\left[(\sin x - \cos x).\frac{(\cos x + \sin x)}{(\sin x - \cos x)} + \log(\sin x - \cos x)\,(\cos x + \sin x)\right]$$

$$\Rightarrow \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}(\cos x + \sin x)[1 + \log(\cos x - \sin x)]$$

10. Differentiate w.r.t. x the function

$x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$.

Solution:

Given function is $x^x + x^a + a^x + a^a$

Let $u = x^x$ and $y = u + x^a + a^x + a^a$ therefore,

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{d}{dx}x^a + \frac{d}{dx}a^x + \frac{d}{dx}a^a$$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + ax^{a-1} + a^x \log a + 0$ ...(i)

Here, $u = x^x$, taking log on both sides

$\log u = \log x^x = x.\log x$

Therefore,

$$\frac{1}{u}\frac{du}{dx} = x.\frac{d}{dx}\log x + \log x.\frac{d}{dx}x = x.\frac{1}{x} + \log x.1 \Rightarrow \frac{du}{dx} = u(1 + \log x) = x^x(1 + \log x)$$

Putting the value of $\frac{du}{dx}$ in equation (i), we get

$$\frac{dy}{dx} = x^x(1 + \log x) + ax^{a-1} + a^x \log a$$

11. Differentiate w.r.t. x the function

$x^{x^2-3} + (x-3)^{x^2}$, for $x > 3$.

Solution:

Given function is $x^{x^2-3} + (x-3)^{x^2}$

Let $u = x^{x^2-3}$ and $v = (x-3)^{x^2}$ therefore, $y = u + v$

Differentiating with respect to x, we have

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...(i)

Here, $u = x^{x^2-3}$, taking log on both sides

$\log u = (x^2 - 3)\log x$, therefore,

$$\frac{1}{u}\frac{du}{dx} = (x^2 - 3).\frac{d}{dx}\log x + \log x.\frac{d}{dx}(x^2 - 3)$$

$$= (x^2 - 3).\frac{1}{x} + \log x.2x$$

$$\frac{du}{dx} = u\left[\frac{x^2 - 3 + 2x^2 \log x}{x}\right]$$

$\frac{du}{dx} = x^{x^2-3}\left[\frac{x^2-3+2x^2\log x}{x}\right] = x^{x^2-4}(x^2 - 3 + 2x^2 \log x)$ ...(ii)

and, $v = (x-3)^{x^2}$, taking log on both sides

$\log v = x^2 \log(x-3)$, therefore,

$$\frac{1}{v}\frac{dv}{dx} = x^2.\frac{d}{dx}\log(x-3) + \log(x-3).\frac{d}{dx}x^2$$

$$= x^2.\frac{1}{x-3} + \log(x-3).2x = \frac{x^2}{x-3} + 2x.\log(x-3)$$

$\frac{dv}{dx} = v\left[\frac{x^2}{x-3} + 2x.\log(x-3)\right] = (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x.\log(x-3)\right]$ ...(iii)

Putting the value of $\frac{du}{dx}$ from (ii) and value of $\frac{dv}{dx}$ from (iii) in equation (i), we have

$$\frac{dy}{dx} = x^{x^2-4}(x^2 - 3 + 2x^2 \log x) + (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x.\log(x-3)\right]$$

12. Find $\frac{dy}{dx}$, if $y = 12(1 - \cos t), x = 10(t - \sin t), -\frac{\pi}{2} < t < \frac{\pi}{2}$.

Solution:

Given that $y = 12(1 - \cos t), x = 10(t - \sin t), -\frac{\pi}{2} < t < \frac{\pi}{2}$

Here, $x = 10(t - \sin t), y = 12(1 - \cos t)$

Therefore, $\frac{dx}{dt} = 10(1 - \cos t)$ and $\frac{dy}{dt} = 12(0 + \sin t)$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12 \sin t}{10(1-\cos t)} = \frac{6\left(2\sin\frac{t}{2}\cos\frac{t}{2}\right)}{5\left(2\sin^2\frac{t}{2}\right)} = \frac{6}{5}\cot\frac{t}{2}$$

13. Find $\frac{dy}{dx}$, if $y = \sin^{-1} x + \sin^{-1}\sqrt{1-x^2}, 0 < x < 1$.

Solution:

Given that $y = \sin^{-1} x + \sin^{-1}\sqrt{1-x^2}, 0 < x < 1$.

Here, $y = \sin^{-1} x + \sin^{-1}\sqrt{1-x^2}$, therefore

$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1} x + \frac{d}{dx}\sin^{-1}\sqrt{1-x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \frac{d}{dx}\sqrt{1-x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{x}.\frac{1}{2\sqrt{1-x^2}}\frac{d}{dx}(1-x^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{x}.\frac{1}{2\sqrt{1-x^2}}(-2x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$$

14. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for $-1 < x < 1$. Prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$

Solution:

Given that $x\sqrt{1+y} + y\sqrt{1+x} = 0 \quad \Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring both sides

$x^2(1+y) = y^2(1+x) \quad \Rightarrow x^2 + x^2y = y^2 + y^2x$

$\Rightarrow x^2 - y^2 + x^2y - y^2x = 0$

$\Rightarrow (x+y)(x-y) + xy(x-y) = 0 \Rightarrow (x-y)(x+y+xy) = 0$

$\Rightarrow (x+y+xy) = 0 \quad [\because x \neq y \ \Rightarrow x - y \neq 0]$

$\Rightarrow y(1+x) = -x$

$\Rightarrow y = -\frac{x}{1+x}$

Therefore,

$$\frac{dy}{dx} = -\left[\frac{(1+x)\frac{d}{dx}x - x\frac{d}{dx}(1+x)}{(1+x)^2}\right] = -\frac{1+x-x}{(1+x)^2} = -\frac{1}{(1+x)^2}$$

15. If $(x-a)^2 + (y-b)^2 = c^2$ for some $c > 0$, prove that

$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ is a constant independent of a and b.

Solution:

Given that $(x-a)^2 + (y-b)^2 = c^2$

Differentiating with respect to x, we have

$$\frac{d}{dx}(x-a)^2+\frac{d}{dx}(y-b)^2=\frac{d}{dx}c^2 \quad \Rightarrow 2(x-a)+2(y-b)\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{x-a}{y-b}$$

Differentiating again, we have

$$\frac{d^2y}{dx^2}=-\frac{(y-b)\frac{d}{dx}(x-a)-(x-a)\frac{d}{dx}(y-b)}{(y-b)^2}=-\frac{(y-b)1-(x-a)\frac{dy}{dx}}{(y-b)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2}=-\frac{(y-b)1-(x-a)\left(-\frac{x-a}{y-b}\right)}{(y-b)^2}=-\frac{(y-b)^2+(x-a)^2}{(y-b)^3}=-\frac{c^2}{(y-b)^3}$$

Putting the values in $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$, we have

$$\frac{\left[1+\left(-\frac{x-a}{y-b}\right)^2\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}}=\frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}}$$

$$=\frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}}=\frac{\left[\frac{c^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}}$$

$=\frac{\frac{c^3}{(y-b)^3}}{-\frac{c^2}{(y-b)^3}}=-\frac{c^3}{c^2}=-c$, which is a constant independent of a and b.

16. If $\cos y=x\cos(a+y)$, with $\cos a\neq\pm1$, prove that $\frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$

Solution:

Given that $\cos y=x\cos(a+y) \quad \Rightarrow x=\frac{\cos y}{\cos(a+y)}$, therefore,

Differentiating with respect to y, we have

$$\frac{dx}{dy}=\frac{\cos(a+y)\frac{d}{dy}\cos y-\cos y\frac{d}{dy}\cos(a+y)}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dx}{dy}=\frac{\cos(a+y)(-\sin y)-\cos y(-\sin(a+y))}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dx}{dy}=\frac{-\sin y\cos(a+y)+\cos y\sin(a+y)}{\cos^2(a+y)}$$

$$=\frac{\sin(a+y-y)}{\cos^2(a+y)}=\frac{\sin a}{\cos^2(a+y)}$$

$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin \alpha}$

17. If $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t\cos t)$, find $\frac{d^2y}{dx^2}$.

Solution:

Given that $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t\cos t)$

Here, $x = a(\cos t + t \sin t), y = a(\sin t - t\cos t)$

Therefore,

$\frac{dx}{dt} = a[-\sin t + (t\cos t + \sin t)] = at\cos t$

and

$\frac{dy}{dt} = a[(\cos t - (-t\sin t + \cos t))] = at\sin t$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{at\sin t}{at\cos t} = \tan t$

$\Rightarrow \frac{d^2y}{dx^2} = \sec^2 t.\frac{dt}{dx} = \sec^2 t.\frac{1}{at\cos t} = \frac{\sec^3 t}{at}$

18. If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x and find it.

Solution:

Given function is $f(x) = |x|^3$

Rewriting the function $f(x) = |x|^3$ in the following form:

$$f(x) = \begin{cases} x^3 & \text{if } x \geq 0 \\ -x^3 & \text{if } x < 0 \end{cases}$$

If $x \geq 0, f(x) = x^3 \quad \Rightarrow f'(x) = 3x^2 \quad \Rightarrow f''(x) = 6x$

If $x < 0, f(x) = -x^3 \quad \Rightarrow f'(x) = -3x^2 \quad \Rightarrow f''(x) = -6x$

Hence, $f''(x)$ exists for all real x and it can be represented as follows:

$$f''(x) = \begin{cases} 6x & \text{if } x \geq 0 \\ -6x & \text{if } x < 0 \end{cases}$$

19. Using mathematical induction prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n.

Solution:

Let $P(n): \frac{d}{dx}(x^n) = nx^{n-1}$

Putting $n = 1$, we have LHS $= \frac{d}{dx}(x^1) = 1$ and RHS $= 1x^{1-1} = x^0 = 1$

Hence, $P(n)$ is true for $n = 1$.

Let $P(k): \frac{d}{dx}(x^k) = kx^{k-1}$ is true.

To prove: $P(k+1): \frac{d}{dx}(x^{k+1}) = (k+1)x^k$ is also true.

$$\text{LHS} = \frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x^k.x) = x^k\frac{d}{dx}(x) + x\frac{d}{dx}x^k$$

$$= x^k.1 + x.kx^{k-1} = (1+k)x^k = \text{RHS}$$

Hence, $P(n)$ is true for $n = k+1$.

Therefore, by the principle of mathematical induction $P(n)$ is true for all natural numbers n.

20. Using the fact that $\sin(A+B) = \sin A\cos B + \cos A\sin B$ and the differentiation, obtain the sum formula for cosines.

Solution:

Given that $\sin(A+B) = \sin A\cos B + \cos A\sin B$,

Differentiating with respect to x, we have

$$\frac{d}{dx}\sin(A+B) = \left(\sin A\frac{d}{dx}\cos B + \cos B\frac{d}{dx}\sin A\right) + \left(\cos A\frac{d}{dx}\sin B + \sin B\frac{d}{dx}\cos A\right)$$

$$\Rightarrow \cos(A+B).\left(\frac{dA}{dx} + \frac{dB}{dx}\right)$$

$$= \left(\sin A(-\sin B)\frac{dB}{dx} + \cos B\cos A\frac{dA}{dx}\right) + \left(\cos A\cos B\frac{dB}{dx} + \sin B(-\sin A)\frac{dA}{dx}\right)$$

$$\Rightarrow \cos(A+B).\left(\frac{dA}{dx} + \frac{dB}{dx}\right)$$

$$= (\cos A\cos B - \sin A\sin B)\frac{dB}{dx} + (\cos A\cos B - \sin A\sin B)\frac{dA}{dx}$$

$$\Rightarrow \cos(A+B).\left(\frac{dA}{dx} + \frac{dB}{dx}\right) = (\cos A\cos B - \sin A\sin B)\left(\frac{dA}{dx} + \frac{dB}{dx}\right)$$

$$\Rightarrow \cos(A+B) = \cos A\cos B - \sin A\sin B$$

21. Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer.

Solution:

Function $f(x) = |x-1| + |x-3|$ is continuous for all real points but not differentiable at two points ($x = 1$ and $x = 3$).

22. If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$ prove that $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Solution:

Given that $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, therefore,

$$\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ \frac{dl}{dx} & \frac{dm}{dx} & \frac{dn}{dx} \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ \frac{da}{dx} & \frac{db}{dx} & \frac{dc}{dx} \end{vmatrix}$$

$$\Rightarrow \frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ 0 & 0 & 0 \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ 0 & 0 & 0 \end{vmatrix}$$

$$\Rightarrow \frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix} + 0 + 0 \quad \Rightarrow \frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

23. If $y = e^{a\cos^{-1}x}, -1 \le x \le 1$, show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

Solution:

Given that: $y = e^{a\cos^{-1}x}$, therefore,

Differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}e^{a\cos^{-1}x} = e^{a\cos^{-1}x}\frac{d}{dx}a\cos^{-1}x$$

$$\Rightarrow \frac{dy}{dx} = e^{a\cos^{-1}x}a.\frac{-1}{\sqrt{1-x^2}} = -\frac{ay}{\sqrt{1-x^2}}$$

Squaring both the sides, we have

$\left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2} \quad \Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2$

Differentiating again with respect to x, we have

$$(1-x^2).2\frac{dy}{dx}.\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\frac{d}{dx}(1-x^2) = a^2 2y\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left[2(1-x^2)\frac{d^2y}{dx^2} + \frac{dy}{dx}(-2x)\right] = 2a^2y\frac{dy}{dx}$$

$$\Rightarrow 2\frac{dy}{dx}\left[(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx}\right] = 2a^2y\frac{dy}{dx}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = a^2y$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$