Mathematics

Single correct answer type:

1. The normal at $(a, 2a)$ on $y^2 = 4ax$, meets the curve again at $(at^2, 2at)$, then the value of t is

(A) -1 (B) 1 (C) -3 (D) 3

Solution: (C)

Let the normal meets the curve again at $(at_2^2, 2at_2)$, then $t_2 = t$.

If the normal at $(at_1^2, 2at_2)$, then

$$t_2 = -t_1 - \frac{2}{t_1}$$

The value of parameter t_1 for the point $(a, 2a)$ is given by $at_1^2 = a$ and $2at_1 = 2a$.

$\Rightarrow \quad t_1 = 1$

$\therefore \quad t_2 = -t_1 - \frac{2}{t_1}$

$\Rightarrow \quad t_2 = -t_1 - \frac{2}{1} = -3$

Hence, $t = -3$

2. The value of $\left[\sqrt{2}\{\cos(56^o 15') + i\sin(56^o 15')\}\right]^8$ is

(A) $-16i$ (B) $16i$ (C) $8i$ (D) $4i$

Solution: (B)

Using De-Moivre's theorem, we have

$$\left(\sqrt{2}\{\cos(56^o\ 15') + i\sin(56^o\ 15')\}\right)^8$$

$= 16\ (\cos 450^o + i \sin 450^o) = 16i$

$= 16(\cos 90^o + i \sin 90^o)$

$= 16(0 + i)$

3. If α is an nth root of unity, then $1 + 2\alpha + 3\alpha^2 + \cdots + n\alpha^{n-1}$ equals

(A) $\frac{n}{(1-\alpha)}$ (B) $-\frac{n}{(1+\alpha)^2}$ (C) $\frac{n}{(1-\alpha)}$ (D) None of these

Solution: (A)

Let $S = 1 + 2\alpha + 3\alpha^2 + \cdots + n\alpha^{n-1}$

Then, $\alpha S = \alpha + 2\alpha^2 + 3\alpha^3 + \cdots + (n-1)\ \alpha^{n-1} + n\alpha^n$

$\therefore \quad S - \alpha S = (1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1}) - n\alpha^n$

$\Rightarrow \quad S(1-\alpha) = \dfrac{\alpha^n - 1}{\alpha - 1} - n\alpha^n$

$\Rightarrow \quad S(1-\alpha) = \frac{1-1}{\alpha-1} - n \qquad [\because \quad \alpha^n = 1]$

$\Rightarrow \quad S = -\dfrac{n}{(1-\alpha)}$

4. If $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, then $(A(adj\ A)A^{-1})A$ is equal to

(A) $2\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & \frac{1}{6} & -\frac{1}{6} \\ \frac{2}{6} & \frac{1}{6} & \frac{3}{6} \\ \frac{3}{6} & \frac{2}{6} & \frac{1}{6} \end{bmatrix}$

(C) $\begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$ (D) None of these

Solution: (A)

We have, $|A| = \begin{vmatrix} 0 & 1 & -1 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix} = 0 + 7 - 1 = 6$

$\therefore \quad (A(adj\ A)A^{-1})\ A = \big(A(adj\ A)\big)(A^{-1}A)$

$= (|A|I)I = |A|I^2$

$= |A|I = 6I = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 2 \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

5. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ and $10B = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$. If B is the inverse of A, then α is

(A) 5 (B) -2 (C) 1 (D) -1

Solution: (A)

WE have,

$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ and $10B = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$

Given that B is inverse of A.

$\therefore \quad AB = I$

$\Rightarrow \quad A(10B) = 10I$ [Multiplying by 10 on both sides]

$\Rightarrow \quad \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} 10 & 0 & 5-\alpha \\ 0 & 10 & \alpha-5 \\ 0 & 0 & \alpha+5 \end{bmatrix} =$

$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$

$\therefore \quad \alpha = 5$

6. If A and B are two matrices such that rank of A = m and rank of B = n, then

(A) $rank\ (AB)\ \geq rank\ (B)$ (B) $rank\ (AB)\ \geq rank\ (A)$

(C) $rank\ (AB)\ \leq \min(rank\ A, rank\ B)$ (D) $rank\ (AB) = mn$

Solution: (C)

We know that,

$rank\ (AB) \leq rank\ (A)$

and $rank\ (AB)\ \leq rank\ \ (B)$

$\therefore \quad rank\ \ (AB) \leq \min(rank\ A, rank\ B)$

7. If the direction cosines of two lines are connected by the equations $l + m + n = 0, l^2 + m^2 - n^2 = 0$, then the angle between the lines is

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{3}$

Solution: (D)

We have, $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$

$\Rightarrow \quad l^2 + m^2 - (-l - m)^2 = 0$

$\Rightarrow \quad lm = 0$

$\Rightarrow \quad l = 0\ \ or\ m = 0$

When $l = 0$, then

$l + m + n = 0$

and $l^2 + m^2 - n^2 = 0$

$\Rightarrow \quad m + n = 0$

and $m^2 - n^2 = 0$

$\Rightarrow \quad m + n = 0$

$\Rightarrow \quad m = -n$

$\therefore \quad \frac{l}{0} = \frac{m}{1} = \frac{n}{-1}$

When $m = 0$, then $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$

$\Rightarrow \quad l+n=0$ and $l^2-n^2=0$

$\Rightarrow \quad l+n=0$

$\Rightarrow \quad l=-n$

$\therefore \quad \frac{l}{1}=\frac{m}{0}=\frac{n}{-1}$

Thus, the direction ratios of two lines are proportional to 0, 1, -1 and 1, 0, -1.

Let θ be the angle between the lines.

Then, $\cos\theta=\frac{1\times0+1\times0+(-1)\,(-1)}{\sqrt{0+1+1}\,\sqrt{1+0+1}}=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}$

8. Given $f(x)=\log\left(\frac{1+x}{1-x}\right)$ and $g(x)=\frac{3x+x^3}{1+3x^2}$, then $fog(x)$ equals

(A) $[f(x)]^3$ (B) $-f(x)$ (C) $3f(x)$ (D) None of these

Solution: (C)

We have,

$$f(x)=\log\left(\frac{1+x}{1-x}\right) \text{ and } g(x)=\frac{3x+x^3}{1+3x^2}$$

$$\therefore \quad fog(x)=f\big(g(x)\big)=f\left(\frac{3x+x^3}{1+3x^2}\right)$$

$$=\log\left(\frac{1+\frac{3x+x^3}{1+3x^2}}{1-\frac{3x+x^3}{1+3x^2}}\right)=\log\frac{(1+x)^3}{(1-x)^3}$$

$$=\log\left(\frac{1+x}{1-x}\right)^3=3\log\left(\frac{1+x}{1-x}\right)=3f(x)$$

9. The equation of the plane which contains the origin and the line of intersection of the planes $r\cdot a=d_1$ and $r\cdot b=d_2$, is

(A) $r\cdot(d_1a+d_2b)=0$ (B) $r\cdot(d_2a-d_1b)=0$

(C) $r\cdot(d_2a+d_1b)=0$ (D) $r\cdot(d_1a-d_2b)=0$

Solution: (B)

Any plane passing through the intersection of the planes $r \cdot a = d_1$ and $r \cdot b = d_2$ is

$(r \cdot a - d_1) + \lambda(r \cdot b - d_2) = 0$ ……..(i)

It will pass through the origin, if

$-d_1 - \lambda d_2 = 0$

$\Rightarrow \quad \lambda = \frac{-d_1}{d_2}$

On substituting the value of λ in Equation (i), we get

$(r \cdot a - d_1) - \frac{d_1}{d_2}(r \cdot b - d_2) = 0$

$\Rightarrow \quad r \cdot (d_2 a - d_1 b) = 0$

Which is the required plane.

10. If from a point $P(a, b, c)$ perpendiculars PA and PB are drawn to YZ and ZX-planes, then the equation of the plane OAB is

(A) $bcx + cay + abz = 0$ (B) $bcx + cay - abz = 0$

(C) $-bcx + cay + abz = 0$ (D) $bcx - cay + abz = 0$

Solution: (B)

Since the coordinates of A and B are (0, b, c) and (a, 0, c), respectively.

The equation of a plane passing through O(0, 0, 0) is

$Px + Qy + Rz = 0$ …….(i)

$\because \quad P \times 0 + Q \times b + R \times c = 0$

and $P \times a + Q \times 0 + R \times c = 0$

$\Rightarrow \quad \frac{P}{bc} = \frac{Q}{ac} = \frac{R}{-ab}$

On substituting the values of P, Q and R in equation (i) we get

$bcx + acy - abz = 0$

11. The order of the differential equation whose general solution is given by $y = (C_1 + C_2)\sin(x + C_3) - C_4 e^{x+C_5}$, is

(A) 2 (B) 3 (C) 4 (D) 5

Solution: (B)

We have

Contains $y = (C_1 + C_2)\sin(x + C_3) - C_4 \,.\, e^{x+C_5}$

$\Rightarrow \quad y = C_6 \sin(x + C_3) - C_4 e^{C_5} \,.\, e^x$

Where, $C_6 = C_1 + C_2$

$\Rightarrow \quad y = C_6 \sin(x + C_3) - C_7 e^x$

Where, $C_4\, e^{C_5} = C_7$

Clearly, the above relation three arbitrary constants. So, the order of the given differential equation is 3.

12. If a set A contains n elements, then which of the following cannot be the number of reflexive relations on the set A?

(A) 2^{n+1} (B) 2^{n-1} (C) 2^n (D) 2^{n^2-1}

Solution: (A)

A relation on set A is a subset of $A \times A$.

Let $a = \{a_1, a_2, a_3, \ldots . a_n\}$. Then, a reflexive relation on A must contain atleast n elements

$$(a_1, a_1), (a_2, a_2) \ldots .. (a_n, a_n).$$

$\therefore$ Number of reflexive relations on A is 2^{n^2-n}.

Clearly, $n^2 - n = n, n^2 - n = n - 1, n^2 - n = n^2 - 1$ have solutions in N but $n^2 - n = n + 1$ does not have solutions in N.

So, 2^{n+1} cannot be the number of reflexive relations on A.

13. The relation on the set $A = \{x : |x| < 3, x \in Z\}$ is defined by $R = \{(x, y) : y = |x|, x \neq 1\}$. Then, the number of elements in the power set of R is

(A) 8 (B) 16 (C) 32 (D) 64

Solution: (B)

We have,

$A = \{x : |x| < 3, x \in Z\} = \{-2, -1, 0, 1, 2\}$

and $R = \{(x, y) : y = |x|, |x \neq -1|\}$

$R = \{(-2, 2), (0, 0), (1, 1), (2, 2)\}$

Clearly, R has four elements. So, the number of elements in power set of R is $2^4 = 16$.

14. Which of the following proposition is a tautology?

(A) $(\sim p \vee \sim q) \wedge (p \vee \sim q)$

(B) $\sim q \wedge (\sim p \vee \sim q)$

(C) $\sim p \wedge (\sim p \vee \sim q)$

(D) $(\sim p \vee \sim q) \vee (p \vee \sim q)$

Solution: (D)

$(\sim p \vee \sim q) \vee (p \vee \sim q)$

$= \sim p \vee \left(\sim q \vee (p \vee \sim q)\right)$

$= \sim p \vee (\sim q \vee \sim q))$

$= \sim p \vee (p \vee \sim q)$

$= (\sim p \vee p) \vee \sim q$

$= t \vee \sim q = t$

15. The combined equation of the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$ is

(A) $2x^2 + 5xy + 2y^2 + 4x + 5y - 2 = 0$

(B) $2x^2 + 5xy + 2y^2 = 0$

(C) $2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$

(D) None of the above

Solution: (C)

Let the equation of asymptotes be

$2x^2 + 5xy + 2y^2 + 4x + 5y + \lambda = 0$ ......(i)

The above equation represents a pair of straight lines.

Therefore, $abc + 2fgh - at^2 - bg^2 - ch^2 = 0$

Here, $a = 2, b = 2, h = \frac{5}{2}, g = 2, f = \frac{5}{2}$

and $c = \lambda,$

$$\therefore \quad 4\lambda + 25 - \frac{25}{2} - 8 - \frac{25}{4}\lambda = 0$$

$$\Rightarrow \quad -\frac{9\lambda}{4} + \frac{9}{2} = 0$$

$$\Rightarrow \quad \lambda = 2$$

On putting the value of λ in equation (i) we get

$2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$

Which is the equation of the asymptotes.

16. A point on the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at a distance equal to the mean of length of the semi-major and semi-minor axes from the centre, is

(A) $\left(\frac{2\sqrt{91}}{7}, \frac{-3\sqrt{91}}{14}\right)$ (B) $\left(\frac{-2\sqrt{105}}{7}, \frac{\sqrt{91}}{14}\right)$

(C) $\left(\frac{-2\sqrt{105}}{7}, \frac{-3\sqrt{91}}{14}\right)$ (D) $\left(\frac{2\sqrt{91}}{7}, \frac{3\sqrt{105}}{14}\right)$

Solution: (D)

Let $P(4\cos\theta, 3\sin\theta)$ be a point on the given ellipse such that its distance from the centre (0, 0) of the ellipse is equal to the mean of the lengths of the semi-major and semi-minor axes, i.e.,

$$OP = \frac{4+3}{2}$$

$$\Rightarrow \quad \sqrt{16\cos^2\theta + 9\sin^2\theta} = \frac{7}{2} \;\Rightarrow\; 7\cos^2\theta + 9 = \frac{49}{4}$$

$$\Rightarrow \quad \cos^2\theta = \frac{13}{26}$$

$$\cos\theta = \pm\sqrt{\frac{13}{28}} \text{ and } \sin\theta = \pm\sqrt{\frac{15}{28}}$$

$$\Rightarrow \quad \cos\theta = \pm\frac{\sqrt{91}}{14} \text{ and } \sin\theta = \pm\frac{\sqrt{105}}{14}$$

Hence, the required points are given by $P\left(\pm\frac{2\sqrt{91}}{7}, \pm\frac{3\sqrt{105}}{14}\right)$.

17. The parametric coordinates of any point on the parabola whose focus is (0, 1) and the directrix is $x + 2 = 0$, are

(A) $(t^2 - 1, 2t + 1)$ (B) $(t^2 + 1, 2t + 1)$

(C) $(t^2, 2t)$ (D) $(t^2 + 1, 2t - 1)$

Solution: (A)

The equation of the parabola is given by

$$\sqrt{(x-0)^2 + (y-1)^2} = \left|\frac{x+2}{\sqrt{1+0}}\right|$$

$$\Rightarrow \quad x^2 + (y-1)^2 = (x+2)^2$$

$$\Rightarrow \quad (y-1)^2 = 4(x+1)$$

The parametric coordinates of any point on this parabola are given by $x + 1 = t^2$ and $y = 1 = 2t$. i.e, $(t^2 - 1, 2t + 1)$.

18. Let a, b and c be positive real numbers. The following system of equations in x, y and z, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and $\frac{-x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ has

(A) Finitely many solutions

(B) No solution

(C) Unique solution

(D) Infinitely many solutions

Solution: (C)

Let $\frac{x^2}{a^2} = X, \frac{y^2}{b^2} = Y$ and $\frac{z^2}{c^2} = Z$

Then, the given system of equations reduces to $X + Y - Z = 1, X - Y + Z = 1$ and $-X + Y + Z = 1$. The coefficient matrix A of the above system of equations is given by

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Clearly, $|A| \neq 0$ So, the given system of equations has a unique solution.

19. One ticket is selected at random from 100 tickets numbered 00, 01, 02,…, 98, 99. If x_1 and x_2 denotes the sum and product of the digits on the tickets, then $P(x_1 = 9/x_2 = 0)$ is

(A) $\frac{1}{50}$ (B) $\frac{2}{19}$ (C) $\frac{19}{100}$ (D) None of these

Solution: (B)

Let the number selected by xy.

Then, $x + y = 9, 0 \leq x, y \leq 9$ and $xy = 0$

$\Rightarrow \quad x = 0, y = 9 \; or \; y = 0, x = 9$

$$\therefore \quad P(x_1 = 9/x_2 = 0) = \frac{P(x_1 = 9 \cap x_2 = 0)}{P(x_2 = 0)}$$

Now, $P(x_2 = 0) = \frac{19}{100}$

and $P(x_1 = 9 \cap x_2 = 0) = \frac{2}{100}$

$$\therefore \quad P\left(\frac{x_1 = 9}{x_2 = 0}\right) = \frac{\frac{2}{100}}{\frac{19}{100}} = \frac{2}{19}$$

20. Range of $f(x) = \sin^{-1} x + \tan^{-1} x + \sec^{-1} x$ is

(A) $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ (B) $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$

(C) $\left(\frac{\pi}{4}, \frac{7\pi}{4}\right)$ (D) None of these

Solution: (B)

Given, $f(x) = \sin^{-1} x + \tan^{-1} x + \sec^{-1} x$

Clearly, the domain of $f(x)$ is $x = \pm 1$.

Thus, the range is $\{f(1), f(-1)\}$, $i.e.,\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$.

21. If $\frac{1}{2}\sin^{-1}\left[\frac{3\sin 2\theta}{5+4\cos 2\theta}\right] = \tan^{-1} x$, then x is equal to

(A) $3\cot\theta$ (B) $3\tan\theta$ (C) $\tan 3\theta$ (D) $\frac{1}{3}\tan\theta$

Solution: (D)

$$\frac{3\sin 2\theta}{5+4\cos 2\theta} = \frac{3 \times \frac{2\tan\theta}{1+\tan^2\theta}}{5+4\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)} = \frac{6\tan\theta}{9+\tan^2\theta}$$

On putting $\tan\theta = 3\tan\phi$, we get

$$\frac{18\tan\phi}{9+9\tan^2\phi} = \frac{2\tan\phi}{1+\tan^2\phi} = \sin 2\phi$$

$$\therefore \quad \frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos\theta}\right) = \tan^{-1} x$$

$$\Rightarrow \quad \frac{1}{2}\sin^{-1}(\sin 2\phi) = \tan^{-1} x$$

$$\Rightarrow \quad \phi = \tan^{-1} x$$

$\Rightarrow \quad \tan^{-1}\left(\frac{1}{3}\tan\theta\right) = \tan^{-1} x$

$\Rightarrow \quad x = \frac{1}{3}\tan\theta$

22. The shortest distance from the plane $12x + y + 3z = 327$ to the sphere $x62 + y^2 + z^2 + 4x - 2y - 6z = 155$ is

(A) 13 (B) 26 (C) 39 (D) $41\frac{4}{13}$

Solution: (A)

The given sphere is

$x^2 + y^2 + z^2 + 4x - 2y - 6z - 155 = 0.$

Its centre is $(-2, 1, 3)$ and radius $= \sqrt{4 + 1 + 9 + 155}$

$= \sqrt{169} = 13$

Therefore, the distance of centre $(-2, 1, 3)$ from the plane $12x + 4y + 3z = 327$ is

$$\frac{|12(-2) + 4(1) + 3(3) - 327|}{\sqrt{144 + 16 + 9}} = 26$$

Hence, the shortest distance $= 26 - 13 = 13$

23. If $\vec{d} = \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$ is a non-zero vector and $\left|(\vec{d} \cdot \vec{c})\,(\vec{a} \times \vec{b}) + (\vec{d} \cdot \vec{a})\,(\vec{b} \times \vec{c}) + (\vec{d} \cdot \vec{b})\,(\vec{c} \times \vec{a})\right| = 0$, then

(A) $|\vec{a}| + |\vec{b}| + |\vec{c}| = |\vec{d}|$ (B) $|\vec{a}| = |\vec{b}| = |\vec{c}|$

(C) $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar (D) None of the above

Solution: (C)

$\vec{d} \cdot \vec{c} = (\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}) \cdot \vec{c}$

$= (\vec{a} \times \vec{b}) \cdot \vec{c} + (\vec{b} \times \vec{c}) \cdot \vec{c} + (\vec{c} \times \vec{a}) \cdot \vec{c}$

$= [\vec{a}\ \vec{b}\ \vec{c}]$

Similarly, $\vec{d} \cdot \vec{a} = \vec{d} \cdot \vec{b} = [\vec{a} \; \vec{b} \; \vec{c}]$

$$|(\vec{d} \cdot \vec{c})(\vec{a} \times \vec{b}) + (\vec{d} \cdot \vec{a})(\vec{b} \times \vec{c}) + (\vec{d} \cdot \vec{b})(\vec{c} \times \vec{a})| = 0$$

$$\Rightarrow \quad [\vec{a}\,\vec{b}\,\vec{c}]\,|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}| = 0$$

$$\Rightarrow \quad [\vec{a}\,\vec{b}\;\vec{c}\,] = 0 \qquad [\because \;\; \vec{d} \text{ is non} - \text{zero}]$$

Hence, $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

24. Let $\vec{u}, \vec{v}$ and $\vec{w}$ be such that $|\vec{u}| = 1, |\vec{v}| = 2$ and $|\vec{w}| = 3$. If the projection of $\vec{v}$ along $\vec{u}$ is equal to that of $\vec{w}$ along $\vec{u}$ and vectors $\vec{v}$ and $\vec{w}$ are perpendicular to each other, then $|\vec{u} - \vec{v} + \vec{w}|$ equals

(A) 2 (B) $\sqrt{7}$ (C) $\sqrt{14}$ (D) 14

Solution: (C)

Given, $\frac{\vec{v} \cdot \vec{u}}{|\vec{u}|} = \frac{\vec{w} \cdot \vec{u}}{|\vec{u}|}$

$$\Rightarrow \quad \vec{v} . \vec{u} = \vec{w} . \vec{u}$$

and $\vec{v} \perp \vec{w}$

$$\Rightarrow \quad \vec{v} \cdot \vec{w} = 0$$

Now, $|\vec{u} - \vec{v} + \vec{w}|^2 = |\vec{u}|^2 + |\vec{v}|^2 + |\vec{w}|^2 - 2\vec{u} \cdot \vec{v} - 2\vec{w} \cdot \vec{v} + 2\vec{u} \cdot \vec{w}$

$$= 1 + 4 + 9 + 0 = 14$$

$$\Rightarrow \quad |\vec{u} - \vec{v} + \vec{w}| = \sqrt{14}$$

25. If $\vec{a}$ and $\vec{b}$ are two vectors, such that $\vec{a} \cdot \vec{b} < 0$ and $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$, then the angle between vectors $\vec{a}$ and $\vec{b}$ is

(A) $\frac{3\pi}{4}$ (B) $\frac{\pi}{4}$ (C) π (D) $\frac{7\pi}{4}$

Solution: (A)

We have, $\left|\vec{a} \cdot \vec{b}\right| = \left|\vec{a} \times \vec{b}\right|$

$\Rightarrow \quad |\vec{a}| \cdot \left|\vec{b}\right| |\cos\theta| = |\vec{a}| \left|\vec{b}\right| \sin\theta$

[where, θ is the angle between $\vec{a}$ and $\vec{b}$]

$\Rightarrow \quad |\cos\theta| = |\sin\theta|$

$\Rightarrow \quad \theta = \frac{\pi}{4}\ or \frac{3\pi}{4} \qquad [as\ \ 0 \le \theta \le \pi]$

But $\vec{a} \cdot \vec{b} < 0$, therefore $\theta = \frac{3\pi}{4}$.

26. If $\alpha + \beta + \gamma = a\delta\ and\ \beta + \gamma + \delta = b\alpha, \alpha\ and\ \delta$ are non-collinera, then $\alpha + \beta + \gamma + \delta$ equals

(A) 0 (B) $a\alpha$ (C) $b\delta$ (D) $(a + b)\gamma$

Solution: (A)

Given, $\alpha + \beta + \gamma = a\delta$ …….(i)

and $\beta + \gamma + \delta = b\alpha$ …….(ii)

On adding δ both the sides in equation (i) we get

$\alpha + \beta + \gamma + \delta = (a + 1)\delta$ …….(iii)

On adding α both the sides in equation (ii), we get

$\alpha + \beta + \gamma + \delta = (b + 1)\alpha$ …….(iv)

From equations (ii) and (iv), we get

$(a + 1)\delta = (b + 1)\alpha$ …….(v)

Since, α is not parallel to δ.

$\therefore \quad a + 1 = 0$ and $b + 1 = 0$

From equation (iii), $\alpha + \beta + \gamma + \delta = 0$

27. The differential equation of the curve for which the initial ordinate of any tangent is equal to the corresponding subnormal

(A) Is linear (B) Is homogeneous of second degree

(C) Has separable variables (D) Is of second order

Solution: (A)

If $y = f(x)$ is the curve, then

$$Y - y = \frac{dy}{dx}(X - x)$$

is the equation of the tangent at (x, y).

Putting $X = 0$, then the initial ordinate of the tangent is $y - x\frac{dy}{dx}$.

$\because$ Subnormal at this point is given by $y\frac{dy}{dx}$.

$$y\frac{dy}{dx} = y - x\frac{dy}{dx} \ or \ \frac{y}{x+y} = \frac{dy}{dx}$$

Which is a homogeneous equation and by rewriting it as

$$\frac{dx}{dy} = \frac{x+y}{y} = \frac{x}{y} + 1 \ \ or \ \frac{dx}{dy} - \frac{x}{y} = 1$$

We see that it is also a linear equation.

28. The solution of the equation $\frac{dy}{dx} = \cos(x - y)$ is

(A) $x + \cot\left(\frac{x-y}{2}\right) = C$ (B) $y + \cot\left(\frac{x-y}{2}\right) = C$

(C) $x + \tan\left(\frac{x-y}{2}\right) = C$ (D) None of these

Solution: (A)

Putting $u = x - y$, we get

$$\frac{du}{dx} = 1 - \frac{dy}{dx}$$

Given equation can be written as

$$1 - \frac{du}{dx} = \cos u \ \Rightarrow \ \ (1 - \cos u) = \frac{du}{dx}$$

$$\Rightarrow \quad \int \frac{du}{1-\cos u} = \int dx + C_1$$

$$\Rightarrow \quad \frac{1}{2}\int cosec^2\left(\frac{u}{2}\right) du = \int dx + C_1$$

$$\Rightarrow \quad x + \cot\left(\frac{u}{2}\right) = C$$

$$\Rightarrow \quad x + \cot\left(\frac{x-y}{2}\right) = C$$

29. The differential equation of all parabolas each of which has a latusrectum 4a and whose axes are parallel to the Y-axis is

(A) Of order 1 and degree 2

(B) Of order 2 and degree 3

(C) Of order 2 and degree 1

(D) Of order 2 and degree 2

Solution: (C)

Equation of the family of parabolas is

$(y-k)^2 = 4a(x-h)$

On differentiating w.r.t x, we get

$$2(y-k)\frac{dy}{dx} = 4a$$

$\Rightarrow \quad (y-k)\frac{dy}{dx} = 2a$ ....(i)

Again, differentiating w.r.t x, we get

$$(y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right) = 0$$

$\Rightarrow \quad 2a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$ [substituting $(y-k)$ from equation (i)]

Hence, the order is 2 and degree is 1.

30. The value of the parameter a such that the area bounded by $y = a^2x^2 + ax + 1$, coordinate axes and the line $x = 1$ attains its least value is equal to

(A) -1 (B) $-\frac{1}{4}$ (C) $-\frac{3}{4}$ (D) $-\frac{1}{2}$

Solution: (C)

Clearly, $a^2x^2 + ax + 1$ is positive for all real values of x.

Area under consideration

$$A = \int_0^1 (a^2x^2 + ax + 1)\, dx$$

$$= \frac{a^2}{3} + \frac{a}{2} + 1 = \frac{1}{6}\,(2a^2 + 3a + 6)$$

$$= \frac{1}{6}\left[2\left(a^2 + \frac{3}{2}a + \frac{9}{16}\right) + 6 - \frac{18}{16}\right]$$

$$= \frac{1}{6}\left[2\left(a + \frac{3}{4}\right)^2 + \frac{39}{8}\right]$$

Which is clearly minimum for $a = -\frac{3}{4}$

31. $\int_0^x |\sin t|\, dt$, where $x \in (2n\pi, (2n+1)\pi), n \in N$, is equal to

(A) $4n - 1 - \cos x$ (B) $4n - \sin x$

(C) $4n - \cos x$ (D) $4n + 1 - \cos x$

Solution: (D)

$$I = \int_0^x |\sin t|\, dt$$

$$= \int_0^{2n\pi} |\sin t|\, dt + \int_{2n\pi}^{x} |\sin t|\, dt$$

$$= 2n \int_0^{\pi} |\sin t|\, dt + \int_{2n\pi}^{x} \sin t\, dt$$

(as x lies in either first or second quadrant)

$$= 2n(-\cos t)_0^{\pi} + (-\cos t)_{2n\pi}^{x} = 4n - \cos x + 1$$

32. Let $I_1 = \int_0^1 \frac{e^x\, dx}{1+x}$ $and\ I_2 = \int_0^1 \frac{x^2\, dx}{e^{x^3}\,(2-x^3)}$. Then, $\frac{I_1}{I_2}$ is equal to

(A) $\frac{1}{3e}$ (B) $3e$ (C) $\frac{e}{3}$ (D) $\frac{3}{e}$

Solution: (B)

Given, $I_1 = \int_0^1 \frac{e^x\, dx}{1+x}, I_2 = \int_0^1 \frac{x^2 dx}{e^{x^3}(2-x^3)}$

In I_2, Put $1 - x^3 = t$

$$\therefore \quad I_2 = \frac{1}{3}\int_1^0 \frac{-dt}{e^{1-t}(1+t)} = \frac{1}{3e}\int_0^1 \frac{e^t\, dt}{1+t} = \frac{1}{3e}\, I_1$$

Now, $\frac{I_1}{I_2} = \frac{I_1}{\frac{1}{3e} I_1} = 3e$

33. $\int_{\frac{5}{2}}^{5} \frac{\sqrt{(25-x^2)^3}}{x^4}\, dx$ is equal to

(A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{3}$ (C) $\frac{\pi}{6}$ (D) $\frac{5\pi}{6}$

Solution: (A)

$$I = \int_{\frac{5}{2}}^{5} \frac{\sqrt{(25-x^2)^3}}{x^4}\, dx$$

Let $x = 5\sin\theta \;\Rightarrow\; dx = 5\cos\theta\ d\theta$

$$\therefore \quad I = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\sqrt{(25 - 25\sin^2\theta)}}{5^4 \sin^4\theta} .5\cos\theta \; d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{5^3\cos^3\theta \; .5\cos\theta}{5^4\sin^4\theta} d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot^2\theta \; (cosec^2\theta - 1) d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot^2\theta \; cosec^2\theta \; d\theta - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot^2\theta \; d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot^2\theta \; cosec^2\theta \; d\theta - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (cosec^2\theta - 1)\, d\theta$$

$$= \left[-\frac{\cot^3\theta}{3} + \cot\theta + \theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left(-0 + 0 + \frac{\pi}{2}\right) - \left(-\frac{3\sqrt{3}}{3} + \sqrt{3} + \frac{\pi}{6}\right) = \frac{\pi}{3}$$

34. If there is an error of K% is measuring the edge of a cube, then the per cent error in estimating its volume is

(A) k (B) $3k$ (C) $\frac{k}{3}$ (D) None of these

Solution: (B)

We know that,

$V = x^3$

and the percent error in measuring x is

$$\frac{dx}{x} \times 100 = k$$

The percent error in measuring volume $= \frac{dV}{V} \times 100$

Now, $\frac{dV}{dx} = 3x^2 \Rightarrow \ dV = 3x^2 dx$

$$\Rightarrow \quad \frac{dV}{V} = \frac{3x^2\, dx}{x^3} = 3\frac{dx}{x}$$

$$\therefore \quad \frac{dV}{V} \times 100 = 3\frac{dx}{x} \times 100 = 3k$$

35. The equation of the tangent to the curve $y = be^{-\frac{x}{a}}$ at the point where it crosses the Y-axis, is

(A) $\frac{x}{a} + \frac{y}{b} = 1$ (B) $\frac{x}{a} - \frac{y}{b} = 1$ (C) $ax + by = 1$ (D) $ax - by = 1$

Solution: (A)

Given equation of the curve

$y = be^{-\frac{x}{a}}$ meets the Y-axis at $(0, b)$.

Again, $\frac{dy}{dx} = be^{-\frac{x}{a}} \left(-\frac{1}{a}\right)$

At $(0, b)$, $\frac{dy}{dx} = be^{0} \left(-\frac{1}{a}\right) = \ -\frac{b}{a}$

Therefore , the required tangent is

$$y - b = \frac{b}{a}(x - 0)$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{a} = 1$$

36. Which of the following function is not differentiable at $x = 1$?

(A) $f(x) = \tan(|x - 1|) + |x - 1|$ (B) $f(x) = \sin(|x - 1|) - |x - 1|$

(C) $f(x) = (x^2 - 1)\ |(x - 1)\ (x - 2)|$ (D) None of the above

Solution: (A)

$(a)\ f(x) = \tan(|x-1|) + |x-1|$

$$f'(1^+) = \lim_{h\to 0} \frac{\tan h + h - 0}{h} = 2$$

$$f'(1^-) = \lim_{h\to 0} \frac{\tan|-h|\ + |-h|}{-h} = \lim_{h\to 0} \frac{\tan h + h}{-h} = -2$$

Hence, $f(x)$ is not differentiable at $x = 1$.

$(b)\ f(x) = \sin(|x-1|) - |x-1|$

$$\therefore \quad f'(1^+) = \lim_{h\to 0} \frac{\sin h - h - 0}{h} = 0$$

$$f'(1^-) = \lim_{h\to 0} \frac{\sin|-h| - |-h|}{-h} = \lim_{h\to 0} \frac{\sin h - h}{-h} = 0$$

Hence, $f(x)$ is differentiable at $x = 1$.

$(c)\ f(x) = (x^2 - 1)\ |(x-1)\ (x-2)|$

$\therefore \quad = (x+1)\ [(x-1)\ |x-1|]\ |x-2|$

Which is differentiable at $x = 1$.

37. The value of $\lim_{x\to\infty} \frac{\left(2^{x^n}\right)^{\frac{1}{e^x}} - \left(3^{x^n}\right)^{\frac{1}{e^x}}}{x^n}$ (where, $x \in N$) is

(A) 0 (B) $n \log n \left(\frac{2}{3}\right)$ (C) $\log n \left(\frac{2}{3}\right)$ (D) Not defined

Solution: (A)

$$L = \lim_{x\to\infty} \frac{\left(2^{x^n}\right)^{\frac{1}{e^x}} - \left(3^{x^n}\right)^{\frac{1}{e^x}}}{x^n}$$

$$= \lim_{x\to\infty} \frac{(3)^{\frac{x^n}{e^x}} \left(\left(\frac{2}{3}\right)^{\frac{x^n}{\wedge x\ e}} - 1\right)}{x^n}$$

Now, $\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{n!}{x} = 0$

[differentiating numerator and denominator n times for L'Hospital's rule]

Hence, $L = \lim_{x\to\infty} 3^{\frac{x^n}{e^x}} . \lim_{x\to\infty} \frac{\left(\left(\frac{2}{3}\right)^{\frac{x^n}{e^x}}-1\right)}{\frac{x^n}{e^x}} \lim_{x\to\infty} \frac{1}{e^x}$

$= 1 \times \log\left(\frac{2}{3}\right) \times 0 = 0$

38. If n integers taken at random are multiplied together, then the probability that the last digit of the product is 1, 3, 7 or 9 is

(A) $\frac{4^n}{5^n}$ (B) $\frac{2^n}{5^n}$ (C) $4^n - \frac{2^n}{5^n}$ (D) None

Solution: (B)

In any number, the last digits can be 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Therefore, the last digit of each number can be chosen in 10 ways. Thus, the exhaustive number of ways in 10^n. If the last digit be 1, 3, 7 or 9,then none of the numbers can be even or end in 0 or 5.

Thus, we have choice of 4 digits viz. 1, 3, 7 or 9 with which each of n numbers should end. So, favourable number of ways is 4^n.

Hence, required probability $= \frac{4^n}{10^n} = \frac{2^n}{5^n}$

39. A bag contains $(n+1)$ coins. It is known that one of these coins shows heads o both sides, whereas the other coins are fair. One coin is selected at random and tossed. If the probability that toss results in heads is $\frac{7}{12}$, then the value of n is

(A) 5 (B) 4 (C) 3 (D) None of these

Solution: (A)

Let E_1 denote the event 'A coin with two heads is selected' and E_2 denote the event 'a fair coin is selected'. Again, let A be the event. 'The toss results in head.' Then,

$$P(E_1) = \frac{1}{n+1}, P(E_2) = \frac{n}{n+1}, P\left(\frac{A}{E_1}\right) = 1$$

and $P\left(\frac{A}{E_2}\right) = \frac{1}{2}$

$\therefore \quad P(A) = P(E_1)\,.\,P\left(\frac{A}{E_1}\right) + P(E_2).\,P\left(\frac{A}{E_2}\right)$

$\Rightarrow \quad \frac{7}{12} = \frac{1}{n+1} \times 1 + \frac{n}{n+1} \times \frac{1}{2} \quad \left[\because \; P(A) = \frac{7}{12}\right]$

$\Rightarrow \quad 12 + 6n = 7n + 7 \quad \Rightarrow \quad n = 5$

40. If A and B are two given events, then $P(A \cap B)$ is

(A) Equal to $P(A) + P(B)$

(B) Equal to $P(A) + P(B) + P(A \cup B)$

(C) Not less than $P(A) + P(B) - 1$

(D) Not greater than $P(A) + P(B) - P(A \cup B)$

Solution: (C)

We have, $P(A \cup B) \leq 1$

$\therefore \quad -P(A \cup B) \geq -1$

$\Rightarrow \quad P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1$

$\Rightarrow \quad P(A \cap B) \geq P(A) + P(B) - 1$

Hence, $P(A \cap B)$ is not less then $P(A) + P(B) - 1$.